Circle Geometry

for High School students

Prof Gerrit Stols

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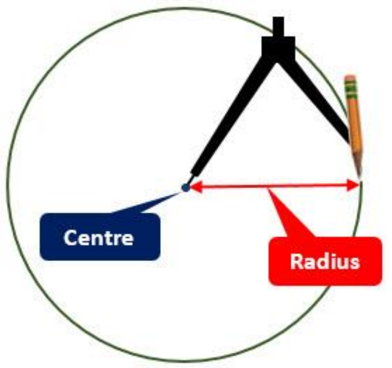
Introduction

Circles are everywhere. We are so used to circles that we do not notice them in our daily lives. In this book, you are about to discover the many hidden properties of circles.

This book will help you to visualise, understand and enjoy geometry. It offers text, videos, interactive sketches, and assessment items. The book will capture the essence of mathematics. Mathematicians are pattern hunters who search for hidden relationships. They also prove and use these relationships to solve problems. In this book you will explore interesting properties of circles and then prove them. Let's just review some important ideas about circles before we continue with the discoveries.

We use a compass to draw a circle.

- The distance between the pin of the compass and pencil is the *radius* of the circle. Any point on the circle is therefore the same distance from the centre.
- The *centre* of the circle is at the sharp point of the compass.



Investigation 0-1: Drag the points and try to describe a diameter, chord and arc of a circle in your own words.

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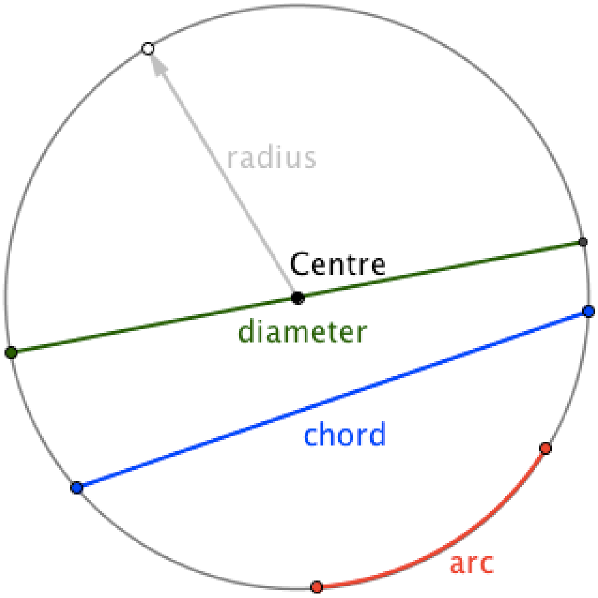


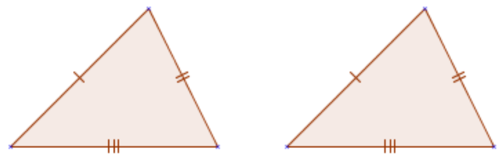
We define a diameter, chord and arc of a circle as follows:
- The distance across a circle through the centre is called the *diameter*. Thus, the diameter of a circle is twice as long as the radius.
- A *chord* of a circle is a line that connects two points on a circle.
- An *arc* is a part of a circle.

You will use results that were established in earlier grades to prove the circle relationships, this include:
- Angles on a straight line add up to 180° (supplementary).
- The angles in a triangle add up to 180°.
- In an isosceles (two equal sides) triangle the two angles opposite the equal sides are themselves equal.
- The exterior angle of a triangle is equal to the sum of interior opposite angles.

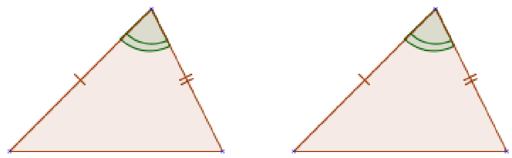
You will often use congruency in proofs. Geometric figures that have the same shape and the same size are congruent. If ΔABC is congruent to ΔDEF we write ΔABC ≡ ΔDEF. As soon as you use this notation the order of the vertices are important. If we move one triangle on top of the other triangle so that all the parts coincide, then vertex A will be on top of vertex D, vertex B will be on top of vertex E, and vertex C will be on top of vertex F.

The minimum conditions for congruency of triangles require three pieces of information:

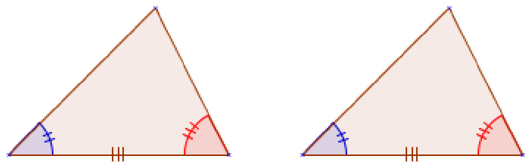
- Side, side, side (SSS): If the three sides of one triangle are equal to the three corresponding sides of the other triangle, then the two triangles are congruent.



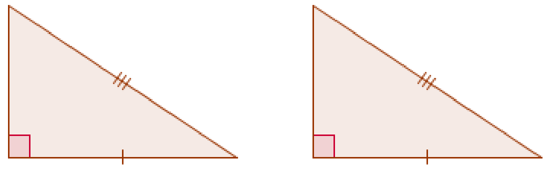
- Side, side, included angle (SAS): If any two sides and the included angle of one triangle are equal to the corresponding two sides and the included angle of the other triangle, the two triangles are congruent.



- Angle, angles & corresponding sides (ASA): If any two angles and a side of one triangle are equal to the corresponding the angles and side of the other triangle, then the two triangles are congruent.



- Right-angle, hypotenuse, side (RHS): If the hypotenuse and a side of one right-angled triangle are respectively equal to the hypotenuse and a side of the other right-angled triangle, then the two triangles are congruent.



THEOREM 1: A LINE FROM THE CENTRE TO A CHORD

Inductive reasoning is what mathematicians use to discover relationships and patterns. We refer to a new discovery as a conjecture. In this section you will explore and discover interesting relationships about perpendicular lines and chords of circles. To express these relationships in your own words you need the following terminology:

- Perpendicular means 90°
- Bisects means to divide into two equal parts
- A chord of a circle is a line that connects two points on the circle.

Investigation 1-1a: This investigation is about a line drawn from the centre to a chord. *Drag the point E.* What do you notice about the length of CE and DE when the angle is 90°? Change the chord by dragging C and D and repeat the process.

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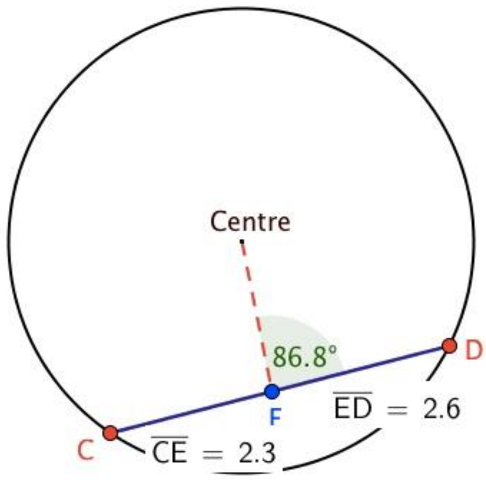


Complete the statement: If the angle is 90° then CE = ...

It is important for you to explore and to try to describe the relationship in your own words before you continue reading. This statement must be expressed in such a way that you can explain your new discovery to somebody over a phone. You need to explain your discovery without referring to this specific sketch. You can use words like circle, centre, segment, chord, bisect, perpendicular.

Conjecture 1a: *If a line is drawn from the centre of a circle perpendicular to a chord, then it bisects the chord.*

Click here for the Interactive

Investigation 1-1b: Drag the point *E* in the previous sketch. What do you notice about the angle if *CE* = *DE*?

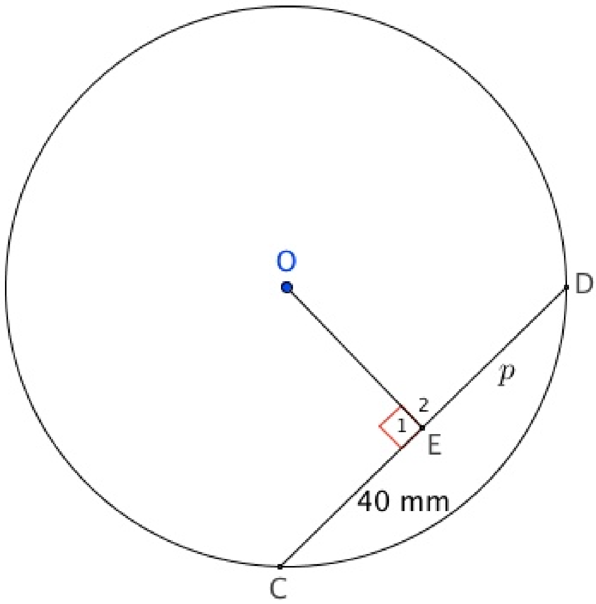
The converse of Conjecture 1a is also true:

Conjecture 1b: *If a line is drawn from the centre of a circle to the midpoint of a chord, then the line is perpendicular to the chord.*

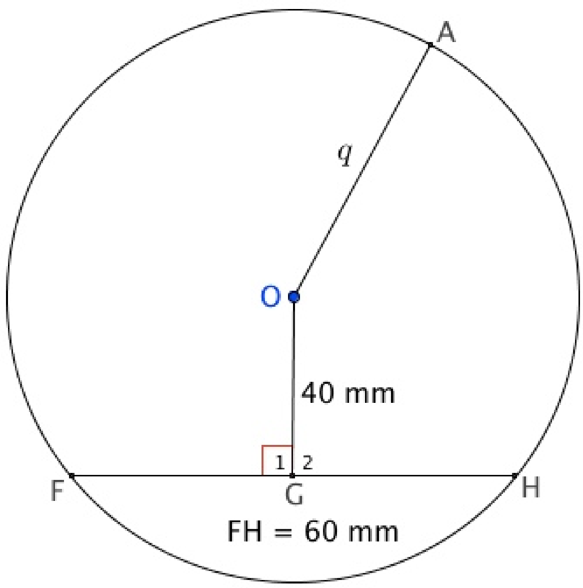
We can use these conjectures about the relationship between the line and the chord to solve problems.

Exercise 1.1

1. Calculate the value of p if O is the centre of the circle.



2. Calculate the value of q if O is the centre of the circle.



PROOF OF CONJECTURE 1

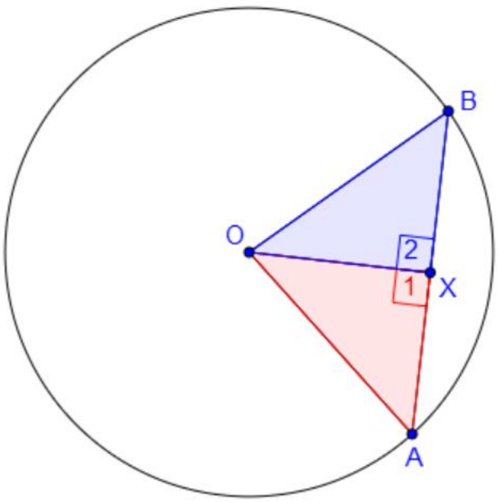
We already discovered and state:
Conjecture 1a: *If a line is drawn from the centre of a circle perpendicular to a chord, then it bisects the chord.*

Is conjecture 1a always valid? As mentioned a mathematician does not accept any result without proof. Note that a proof for the statement "if A is true then B is also true" is an attempt to verify that B is a logical result of having assumed that A is true. You will have to discover the linking relationship between A and B.

Can you think of a way to prove the conjecture? There are different ways to prove the conjecture: you can use *congruency* of triangles or the *Pythagoras theorem*.

The following proof of Conjecture 1a is based on congruency of triangles:

Construction: Connect OA and OB.
Strategy: If we can show that ΔAXO and ΔBXO are congruent then the sides AX and BX must be equal.
Given: OX⊥AB
Required to prove: AX = XB



Proof:
In ΔAOX and ΔBOX is:

OX = OX	(common)
OA = OB	(radii)
$\hat{X}_1 = \hat{X}_2 = 90°$	(given)
Then ΔAOX ≡ ΔBOX	(90°, hypotenuse, side)
Therefore AX = XB	(congruency)

After you manage to prove your conjecture, it will become a theorem:

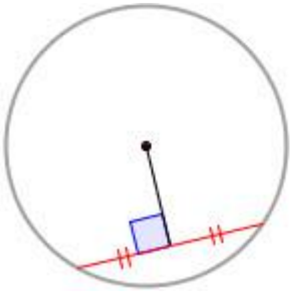
Theorem 1a: If a line is drawn from the centre of a circle perpendicular to a chord, then it bisects the chord.

How will you prove Conjecture 1b which states that if a line is drawn from the centre of a circle to the midpoint of a chord, then the line is perpendicular to the chord?

REVIEW

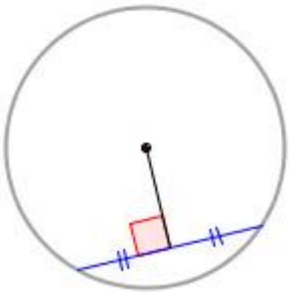
The conjectures that were proved are called theorems and can be used in future proofs. In this lesson you discovered and proved the following:

Theorem 1a: If a line is drawn from the centre of a circle perpendicular to a chord, then it bisects the chord.



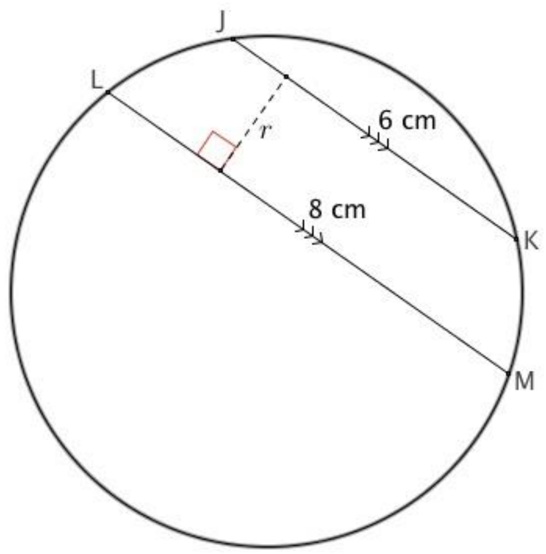
The converse of this theorem:

Theorem 1b: If a line is drawn from the centre of a circle to the midpoint of a chord, then the line is perpendicular to the chord.

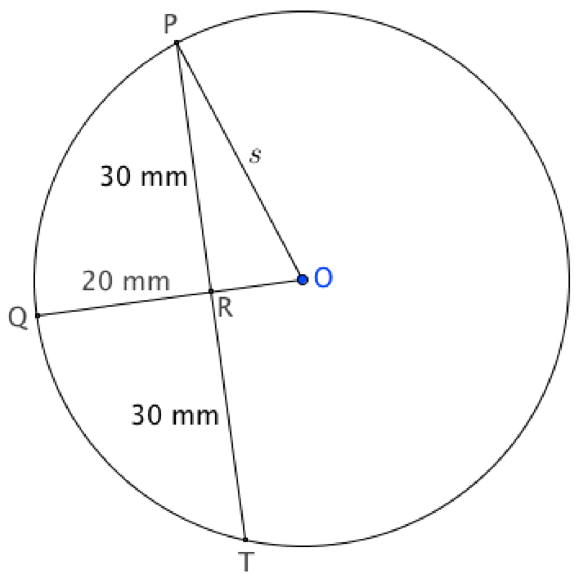


Exercise 1.2

1. Calculate the value of r if the radius of the circle is 5 cm.

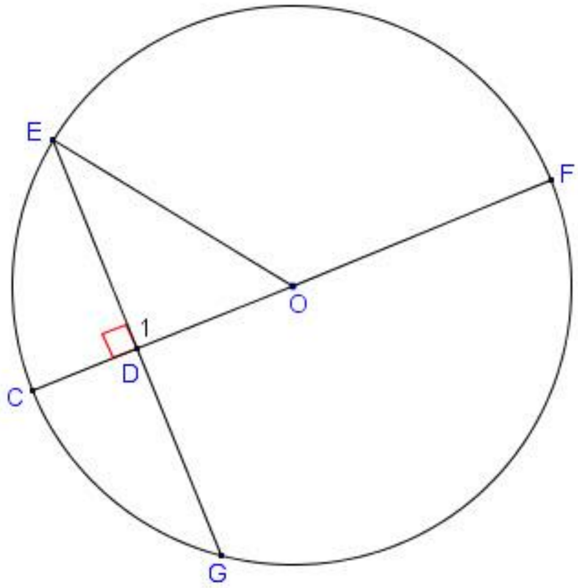


2. Calculate the value of s if O is the centre of the circle.

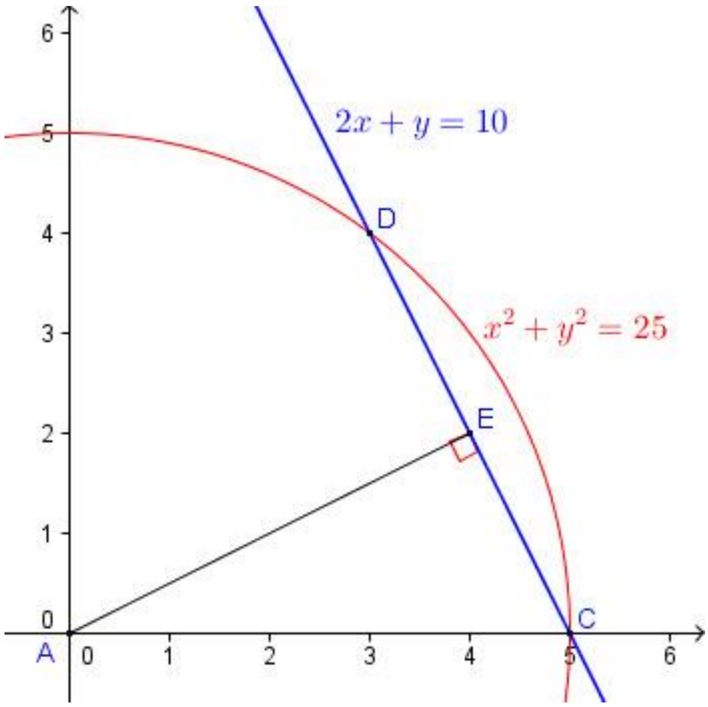


3. Prove Conjecture1b: If a line is drawn from the centre of a circle to the midpoint of a chord, then the line is perpendicular to the chord.

4. EG is a chord of the circle with centre O. *CF* is perpendicular to *EG*. *DF* = 4*DC* and *CD* = 3 cm. Calculate the length of *EG*.



5. The equation of the line is $2x + y = 10$ and that of the circle $x^2 + y^2 = 25$. If $AE \perp DC$ calculate the length of *DE*.



THEOREM 2: THE PERPENDICULAR BISECTOR OF A CHORD

The following investigation is about the perpendicular bisector of a chord. A *perpendicular bisector* is a perpendicular line that passes through the midpoint of a line segment. You will also remember that a *chord* of a circle is a line that connects two points on the circle. To state conjectures in mathematical words you need to know that perpendicular means 90° and that bisects means to divide into two equal parts.

Investigation 2-1: What do you notice about the perpendicular bisector when both of the red points are on the circle (when the red line is a chord)?

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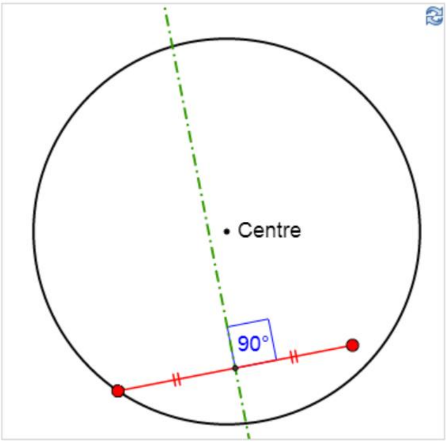

We refer to a new discovery as a conjecture.

Conjecture 2: *If the perpendicular bisector of a chord is drawn, then it passes through the centre of the circle.*

APPLICATIONS

We can use this theorem to locate the centre of any circle. Just construct two chords and their perpendicular bisectors.

Investigation 2-2: Drag the chords. What do you notice about the intersection of the perpendicular bisectors?

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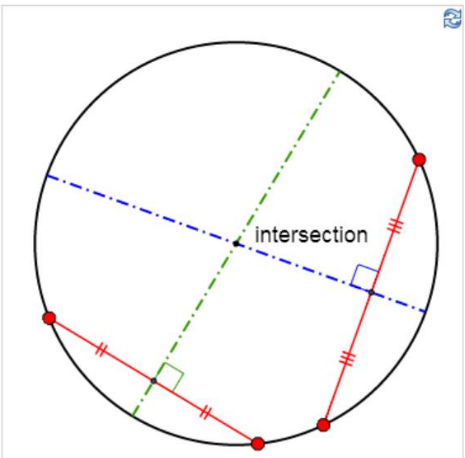

Corollary: The centre of a circle is on the perpendicular bisector of any chord, therefore their intersection point is the centre.

The conjecture also explains why we use perpendicular bisectors if we want to construct a circle circumscribed about a triangle.

Investigation 2-3: Drag the vertices of the triangle, what do you notice about the intersection of the bisectors?

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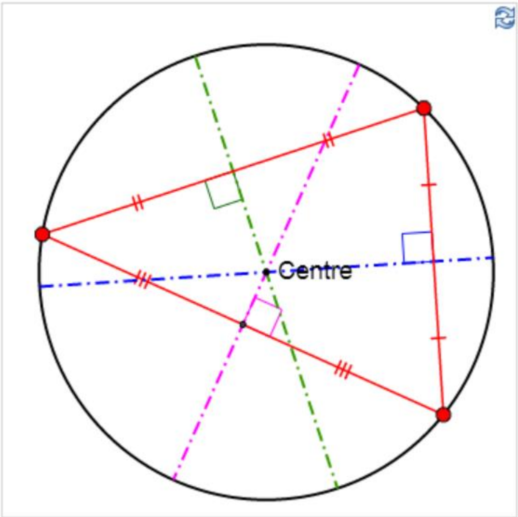


Corollary: The perpendicular bisectors of the sides of a triangle meet at the centre of the circumscribed circle.

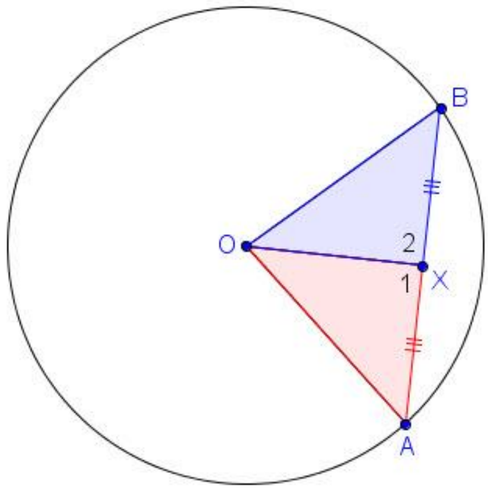
The triangle consists of three chords. According to Theorem 2 the centre of the circle should be on the perpendicular bisectors of all three chords (sides) of the triangle.

Proof of Conjecture 2

We already discovered and stated Conjecture 2: *If the perpendicular bisector of a chord is drawn, then it passes through the centre of the circle.* But as mentioned a mathematician does not accept any result without proof. Note that a proof for the statement **"**if A is true then B is also true**"** is an attempt to verify that B is a logical result of having assumed that A is true. You will have to discover the linking relationship between A and B.

The first step in any proof is to recognise the statements A and B. In general, everything after the word "if" and before the word, 'then' or the comma, is statement A, the premise or everything that you assume to be true. Everything after the word "then" is the statement "B" or the conclusion that you have to prove.

Strategy: We will show that if we construct any random chord the centre is on the perpendicular bisector of the chord. Let O be the centre of the circle. Draw any chord AB and let X be the midpoint of the chord: that is AX = BX.
Construction: Construct OA, OD, and OB.
Given: AX = BX and $\hat{X}_1 = \hat{X}_2 = 90°$



Proof:
Assume that OX does not pass through the perpendicular bisector of a chord then $\hat{X}_1 \neq \hat{X}_2 \neq 90°$.
In ΔAOX and ΔBOX is:

OX = OX	(common)
OA = OB	(radii)
AX = BX	(construction)
Therefore ΔAOX ≡ ΔBOX	(side, side, side)
Then $\hat{X}_1 = \hat{X}_2$	
Therefore $\hat{X}_1 = \hat{X}_2 = 90°$	(angles on a straight line)

Then OX is a perpendicular bisector of chord AB with the centre O on it.

We can therefore state a new theorem after the conjecture has been proved:

Theorem 2: If the perpendicular bisector of a chord is drawn, then it passes through the centre of the circle.

Exercise 2.1

---

1. Points A(2; -6), B(16; 8) and C(2; 14) is on a circle. Find the coordinates of the centre of this circle.

THEOREM 3: ANGLE SUBTENDED BY AN ARC OR CHORD

You will remember that:

- A *chord* of a circle is a line that connects two points on a circle.
- An *arc* is a part of a circle.
- An angle P is *subtended* (or created) by the end points of a line segment AB (or any two points A and B) is the angle ∠APB or $A\hat{P}B$.

This investigation is about the relationship between an angle subtended by an arc or chord at the centre of a circle and the angle subtended by the same arc at the circle.

Investigation 3-1: Drag the points. What do you notice about the sizes of the angles at the centre and on the circle? Describe this relationship in your own words (make a conjecture).

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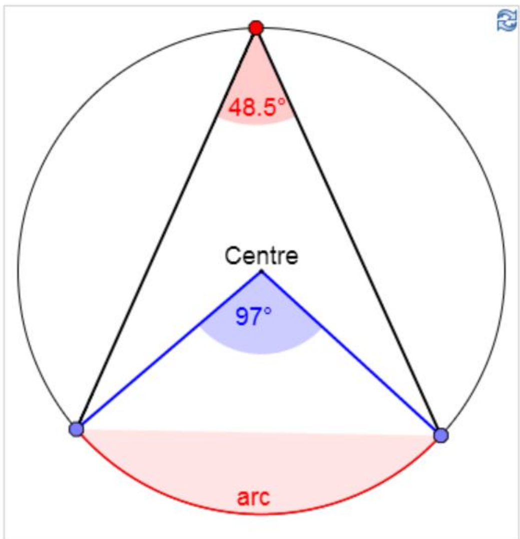

***Conjecture 3:** The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle.*

It is important to notice that the angle on the circle must be on the same side of the chord as the centre.

From Conjecture 3 we know that the angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle. What do you think is the angle on the circle if the angle at the centre is 180°.

Investigation 3-2: Drag the points until the angle at the centre is 180°. What did you notice about the sizes of the angle on the circle? Describe this relationship in your own words (make a conjecture).

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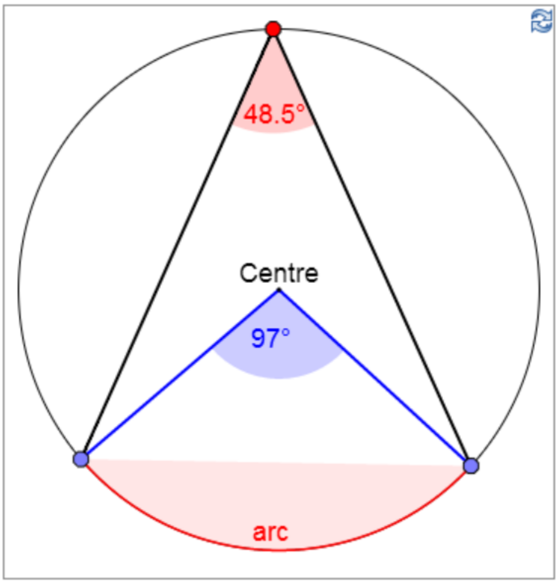

A diameter is the distance across a circle through the centre.

Drag the point on the circle to confirm your finding:

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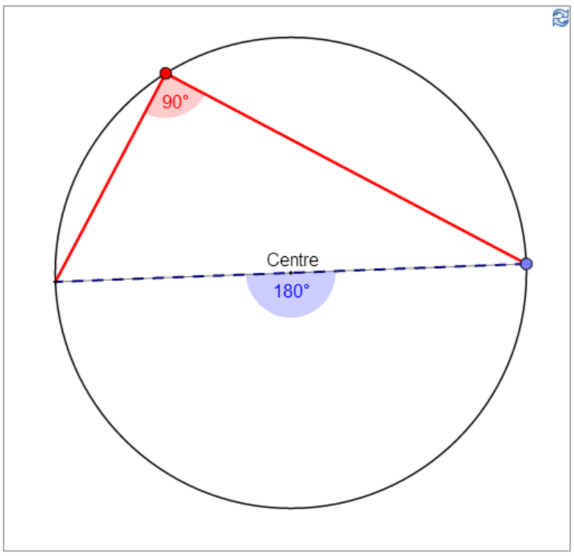

This discovery is named after Thales of Miletus who lived 624- 546 BC.

Thales' theorem: *The angle subtended on the circle by the diameter is always 90°.*

Investigation 3-3: Drag the point X inside, outside and on the circle. What did you notice about the angle size at point X? Describe this relationship in your own words (make a conjecture).

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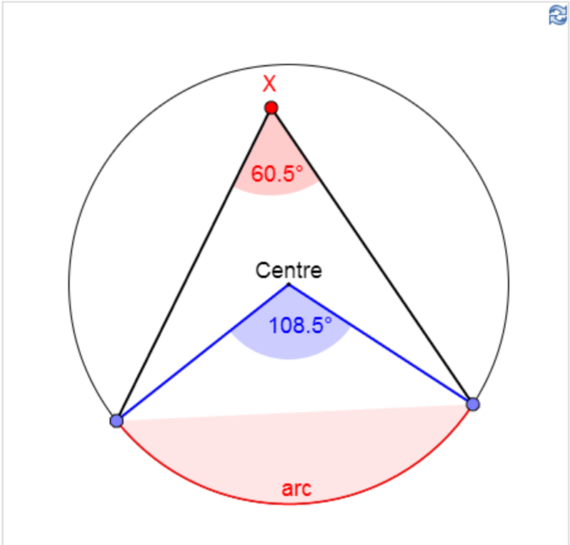

Exercise 3.1

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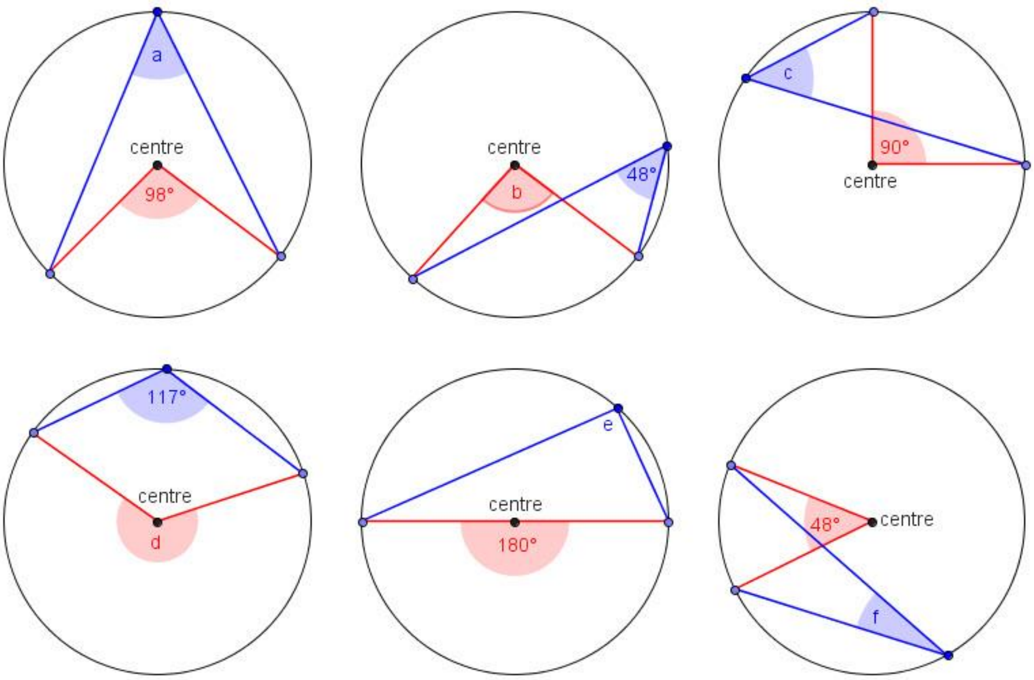
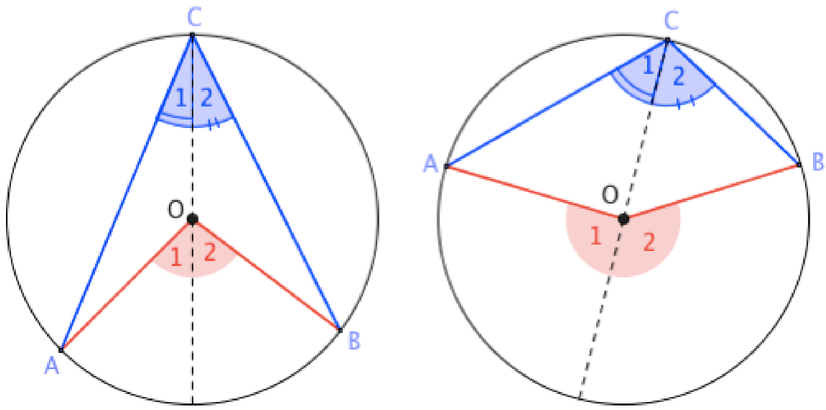
1. Calculate the size of *a, b, c, d, e* and *f*.



PROOF OF CONJECTURE 3

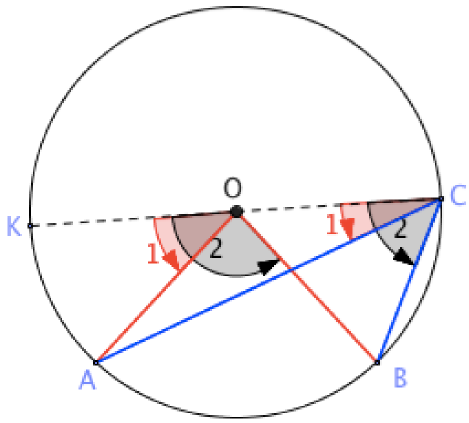
Conjecture 3: *The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle.*

Given: Circle with centre O and arc AB subtending $A\hat{O}B$ at the centre and $A\hat{C}B$ at the circle.
Construction: Draw a line through O and C.
Required to prove: $A\hat{O}B = 2A\hat{C}B$
The proof of conjecture 3 is also tricky because there is more than one possible case.
Case 1: Let $\hat{C}_1$= α and $\hat{C}_2$ = β



$\hat{A} = \hat{C}_1$= α (Base angles of isosceles ΔOAC; OA = OC = radius)
Hence $\hat{O}_1 = 2\alpha$ (External angle of a triangle is equal to the sum of the opposite interior angles)
Similarly $\hat{O}_2 = 2\beta$
Then $A\hat{O}B$ = 2α + 2β = 2(α+β) = $2A\hat{C}B$

Case 2: Let $\hat{C}_1$ = α and $\hat{C}_2 = K\hat{C}B$ = β



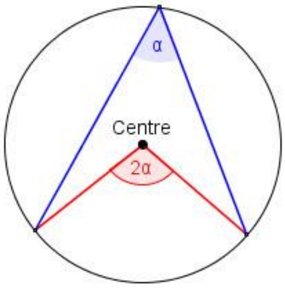
$\hat{A} = \hat{C}_1$ = α (Angles of isosceles ΔOAC; OA = OC = radius)
Hence $\hat{O}_1 = 2\alpha$ (External angle of a triangle = sum of interior angles)
Similarly if $\hat{O}_2 = K\hat{O}B$ then $\hat{O}_2 = 2\beta$
Then $A\hat{O}B = \hat{O}_2 - \hat{O}_1$ = 2α – 2β = 2(α–β) = 2 ($\hat{C}_2 - \hat{C}_1$) = $2A\hat{C}B$
Therefore $A\hat{O}B = 2A\hat{C}B$

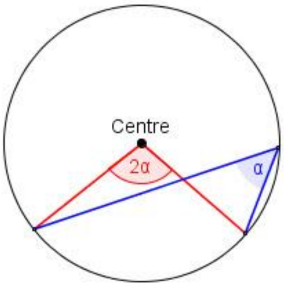
A conjecture is a proposition that is not yet proved. After providing a proof for a conjecture, we call the conjecture a theorem.

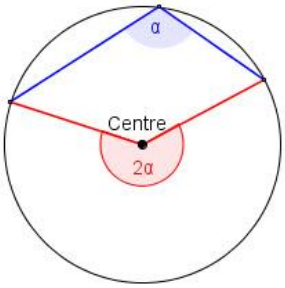
Theorem 3: The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)

REVIEW

Theorem 3: The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)

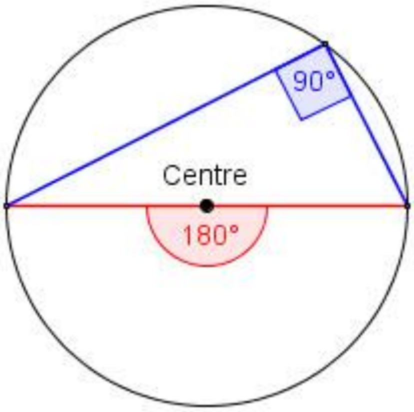






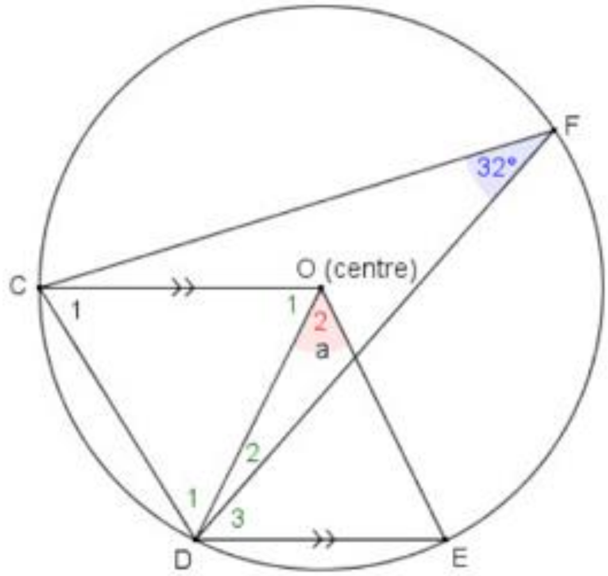
A special case of Theorem 3: Thales' theorem

- The angle subtended on the circle by the diameter is always 90º; or
- An angle in a semicircle is always a right angle.

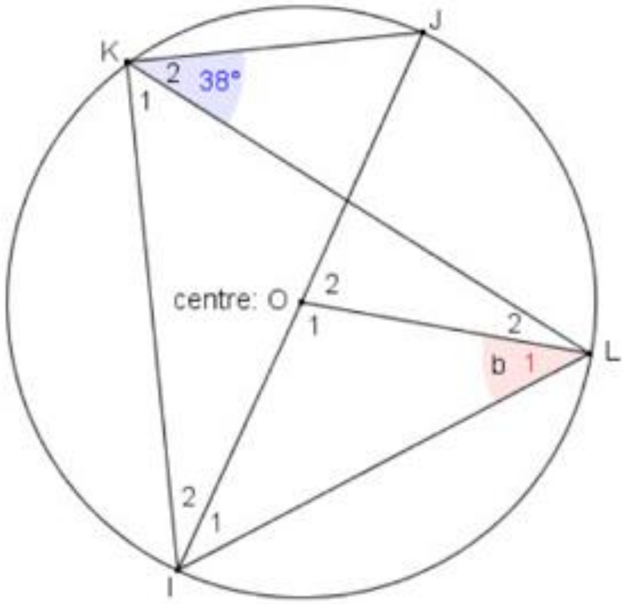


__Exercise 3.2__

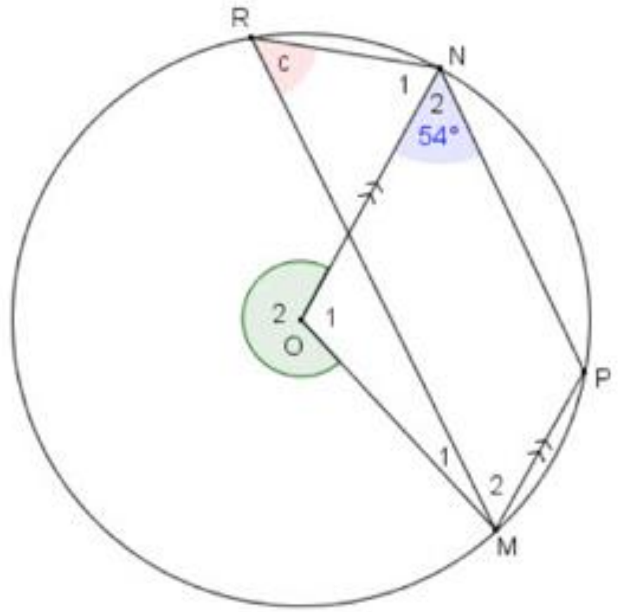
1. O is the centre of the circle. Calculate angle *a*.



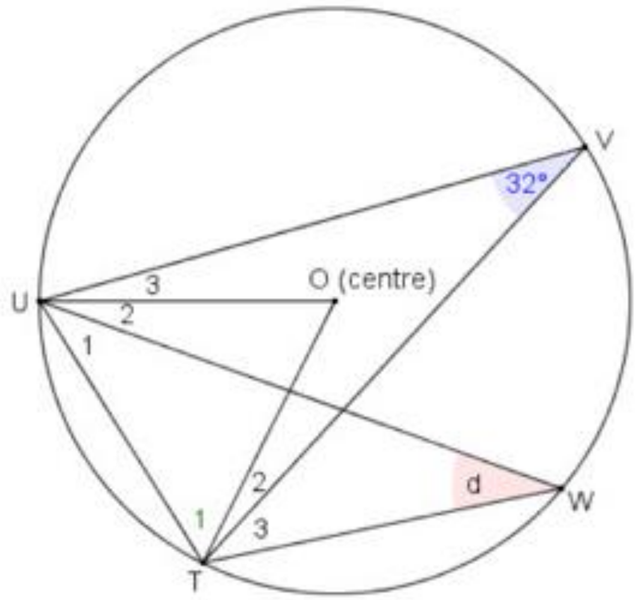
2. O is the centre of the circle. Calculate angle *b*.



3. O is the centre of the circle. Calculate angle *c*.



4. O is the centre of the circle. Calculate angle *b*.

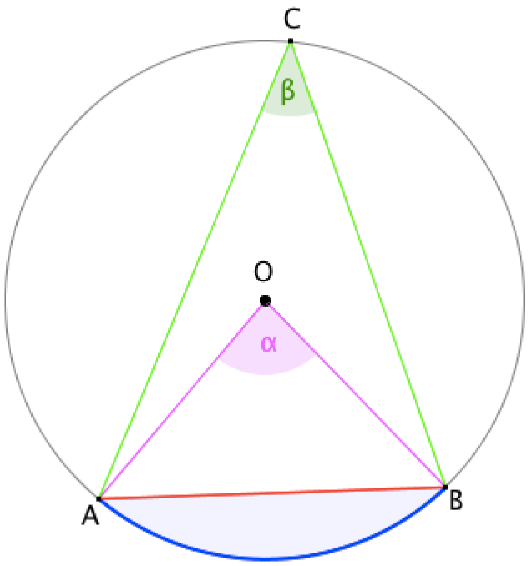


5. Prove Thales' Theorem.

6. Use Thales' theorem to construct a right angle.

THEOREM 4: ANGLES IN A CIRCLE

An arc is a part of a circle and the associated chord is a line segment joining the endpoints of the arc. An angle subtended by an arc or chord is one whose two rays pass through the endpoints of that arc or chord. For example in the accompanying figure arc or chord AB subtends both of the angles $A\hat{O}B = \alpha$ and $A\hat{C}B = \beta$



The next investigation is about the relationship between angles subtended by an arc or chord on a circle.

Investigation 4-1: Drag the points; what do you notice about the sizes of the angles on the circle? Describe this relationship in your own words (make a conjecture).

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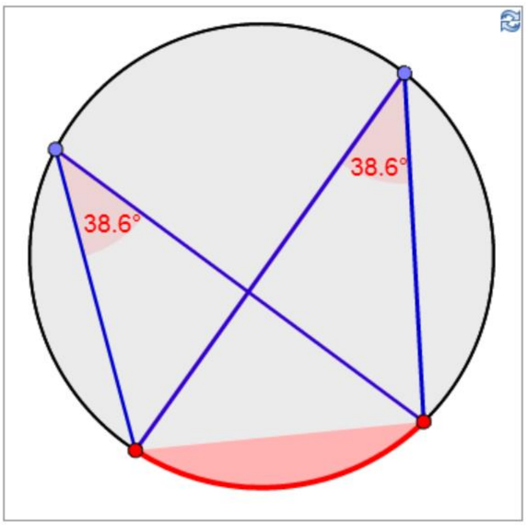

Conjecture 4a: *If angles on a circle are subtended by the same chord or arc THEN the two angles are equal.*

It is important to notice that the angle on the circle must be on the same side of the chord. Why do you think does the relationship hold?

Investigation 4-2: The converse of Conjecture 4: Drag the two angles until they are equal; what do you notice? Make a conjecture.

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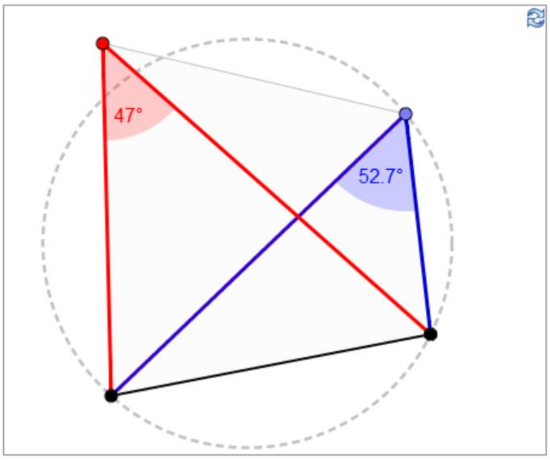

***Conjecture 4b:** If a line segment joining two points subtends equal angles at two other points on the same side of a line segment, these four points are on the same circle.*

Points that are on a circle are called *concyclic*. A related conjecture is about angles subtended by equal chords or arcs:

***Corollary:** Angles subtended by equal chords or arcs have the same size.*

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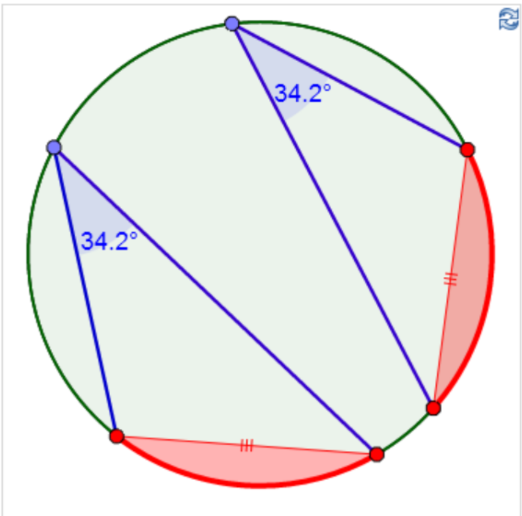
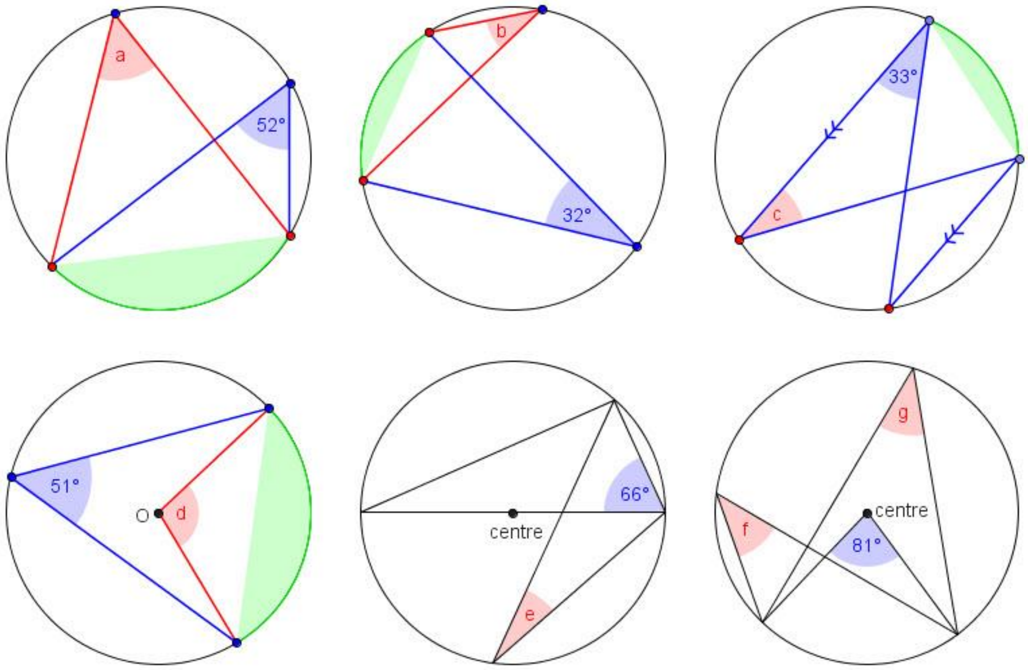


A corollary is just a proposition that follows from one already proved.

Exercise 4.1

Use the given information to calculate the size of the red angle in each case:

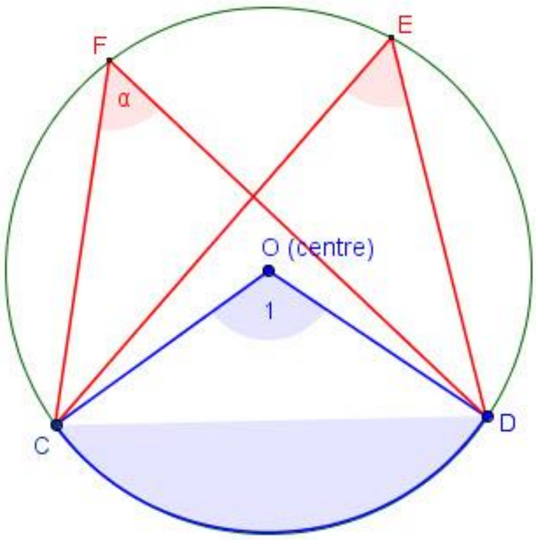


PROOF OF CONJECTURE 4

The proof of conjecture 4 is straightforward. In fact, you have used the idea behind this proof to solve the last question of the previous exercise. It follows from Theorem 3.

Conjecture 4: *Angles subtended by a chord of the circle, on the same side of the chord, are equal.*

Given: Circle with centre O and a chord CD subtending $C\hat{F}D$ and $C\hat{E}D$ in the same segment. Let $\hat{F}$ = α.
Construction: Draw CO and DO
Required to prove: $\hat{E} = \hat{F}$



Proof:

$\hat{O}_1 = 2\hat{F} = 2\alpha$ (Angle at the centre = 2 × angle on the circle)

$\hat{E} = \frac{1}{2}\hat{O}_1 = \frac{1}{2}(2\alpha) = \alpha$ (Angle at the centre = 2 × angle on the circle)

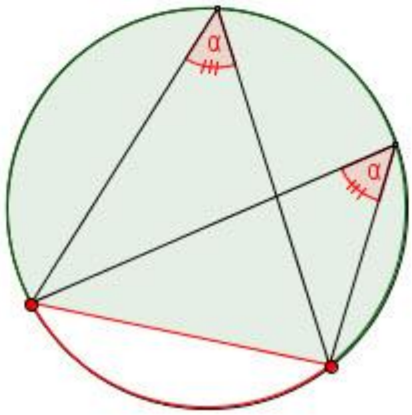
Hence $\hat{E} = \hat{F} = \alpha$

After proving Conjecture 4 we can use it in future proofs and state the following theorem:

Theorem 4: Angles subtended by a chord of the circle, on the same side of the chord, are equal.

REVIEW

Theorem 4: Angles subtended by a chord of the circle, on the same side of the chord, are equal.



Exercise 4.2

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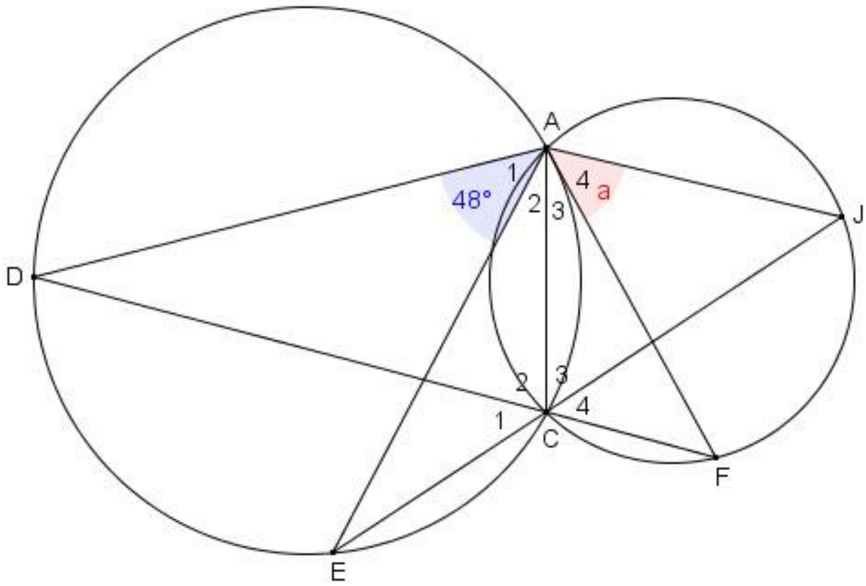
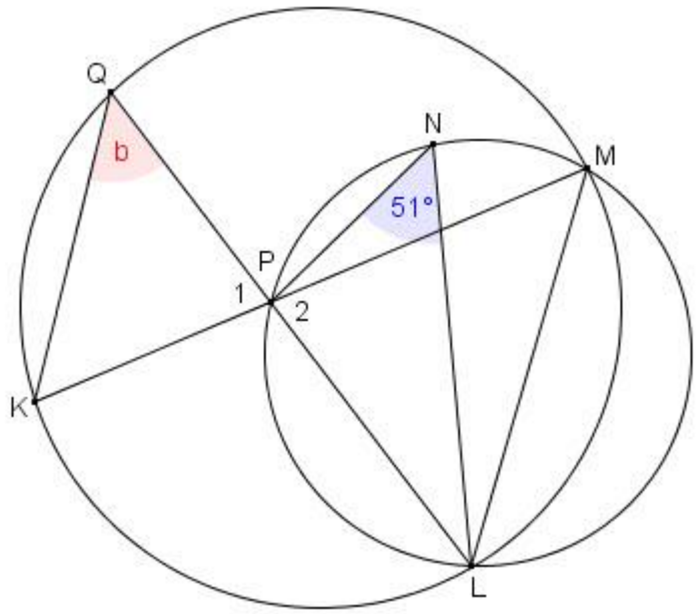
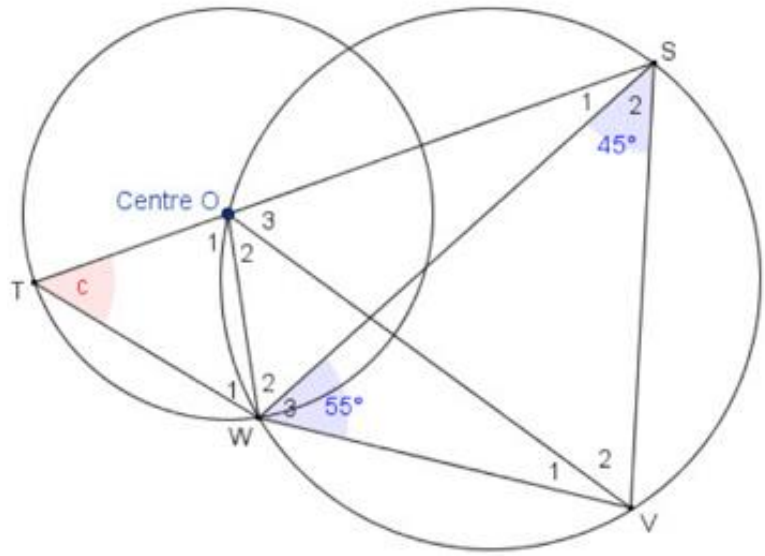
1. Calculate the size of *a*.



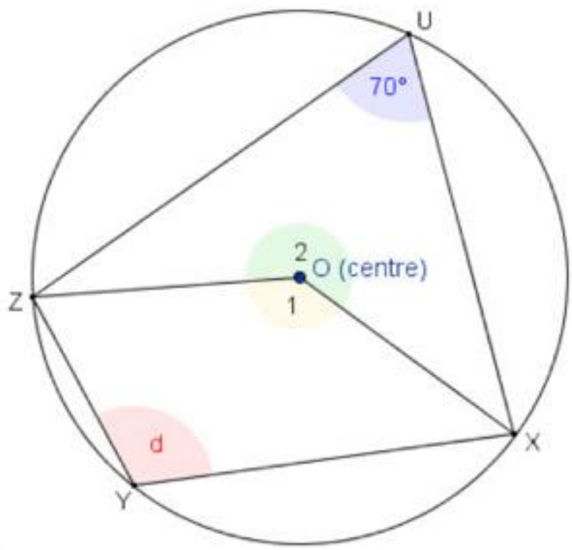
2. Calculate the size of *b*.



3. Calculate the size of *c*.



4. Calculate the size of d.



THEOREM 5: CYCLIC QUADRILATERALS

You already know that:

- *Quadrilateral* is a polygon with four sides and four vertices.
- *Supplementary* means that the sum of the angles is 180°.
- *Cyclic* means that the vertices are on a circle.

In this section we will investigate properties of cyclic quadrilaterals, that is, a quadrilateral with all four vertices on a circle.

Investigation 5-1a: Drag the vertices onto the circle; what do you notice about the size of the two opposite angles? Make a conjecture.

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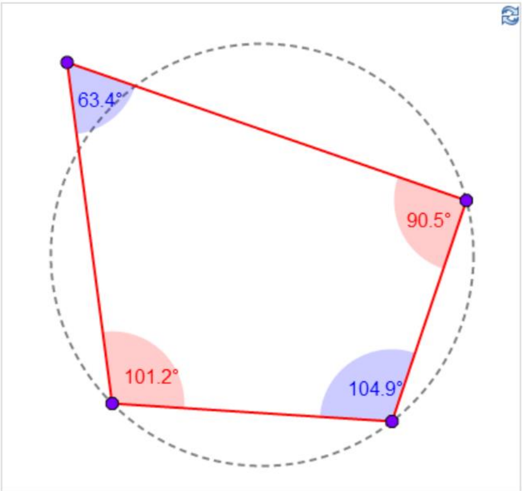


Two angles are supplementary if they add to 180°.

***Conjecture 5a**: If a quadrilateral is cyclic, then the opposite angles are supplementary.*

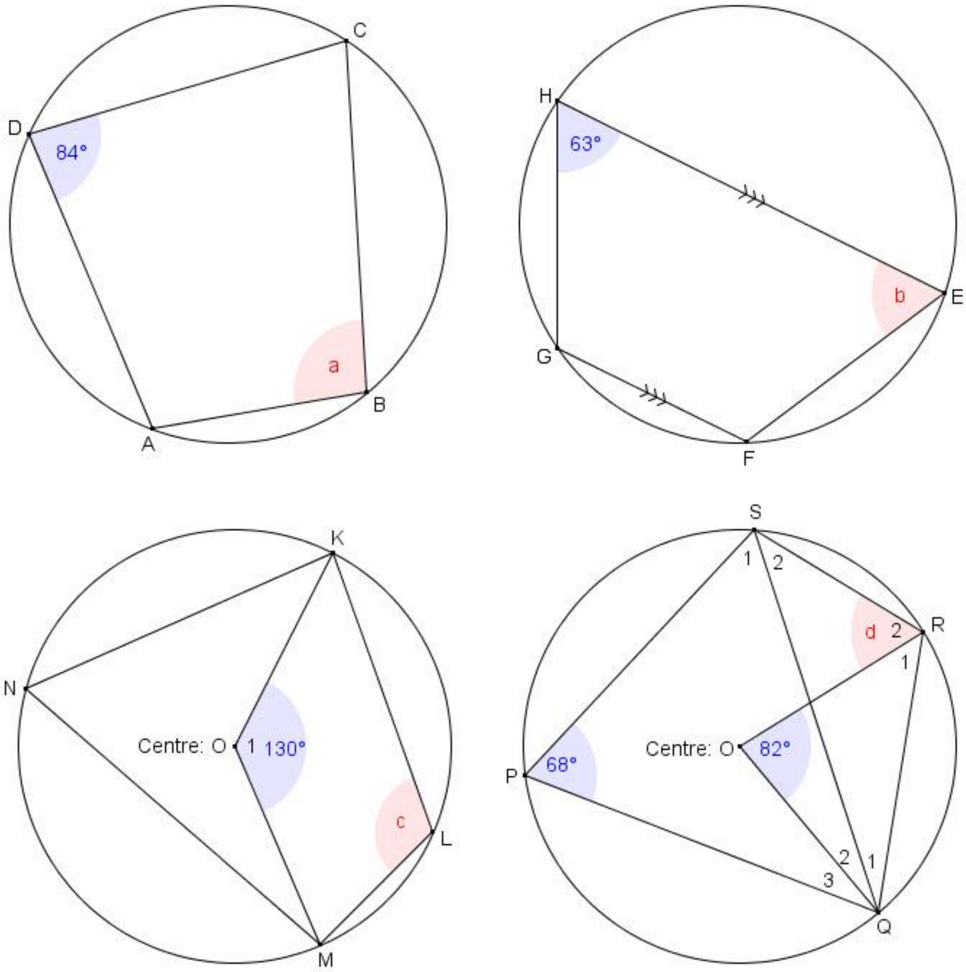
Click here for the Interactive

Investigation 5-1b: You can use the previous sketch to investigate the converse of the conjecture. Drag the vertices until the opposite angles are supplementary; what do you notice?

***Conjecture 5b:** If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.*

Exercise 5.1

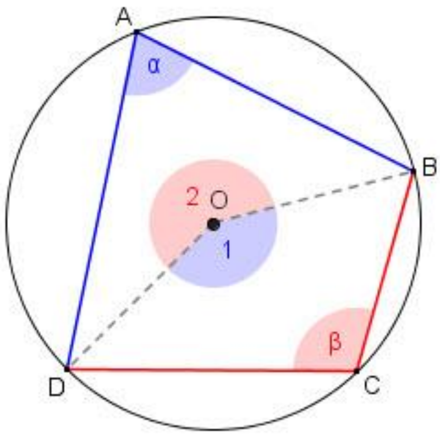
1. Calculate the size of *a, b, c* and *e*.



PROOF OF CONJECTURE 5A

Conjecture 5a: *If a quadrilateral is cyclic, then the opposite angles are supplementary.*

Given: Circle O containing cyclic quadrilateral ABCD. Let $\hat{A} = \alpha$ and $\hat{C} = \beta$
Construction: Draw OB and OD.
Required to prove: $\hat{A} + \hat{C} = \alpha + \beta = 180°$, $\hat{B} + \hat{D} = 180°$

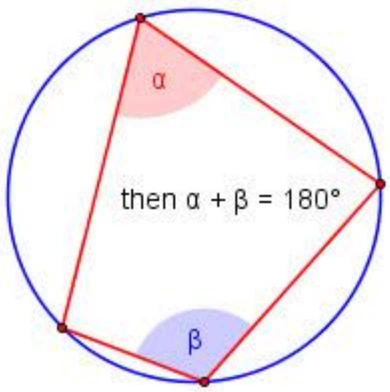


Proof
$\hat{O}_1 = 2\hat{A} = 2\alpha$ & $\hat{O}_2 = 2\hat{C} = 2\beta$ (Angles at centre 2 × angle at circumference)
but $\hat{O}_1 + \hat{O}_2 = 2(\alpha + \beta)$
Hence $2\alpha + 2\beta = 360°$ (Angles around a point)
Then $\hat{A} + \hat{C} = \alpha + \beta = 180°$
We also know that $\hat{B} + \hat{D} = 180°$ because the sum of the interior angles of a quadrilateral is 360°

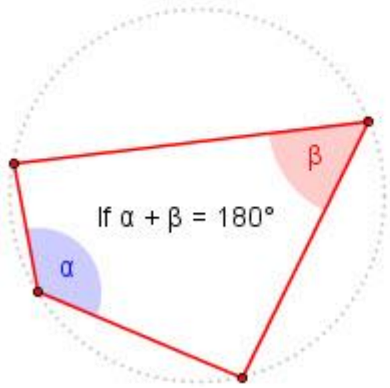
Theorem 5: The opposite angles of a cyclic quadrilateral are supplementary.

REVIEW

Theorem 5a: The opposite angles of a cyclic quadrilateral are supplementary.



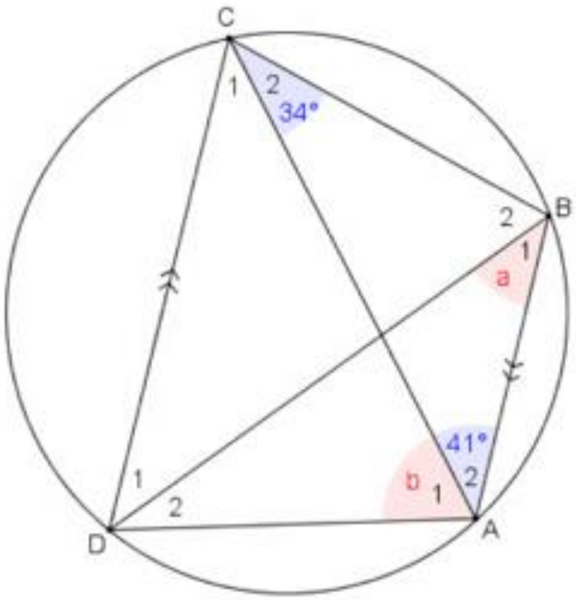
Theorem 5b: If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.



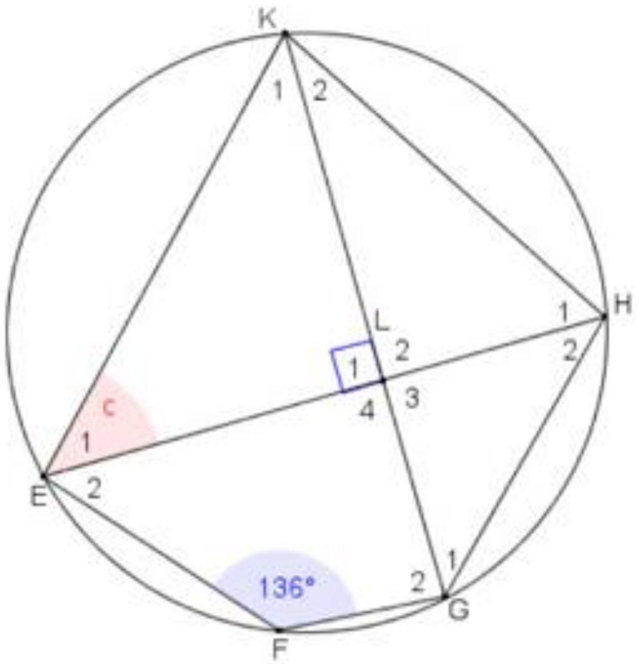
Exercise 5.2

---

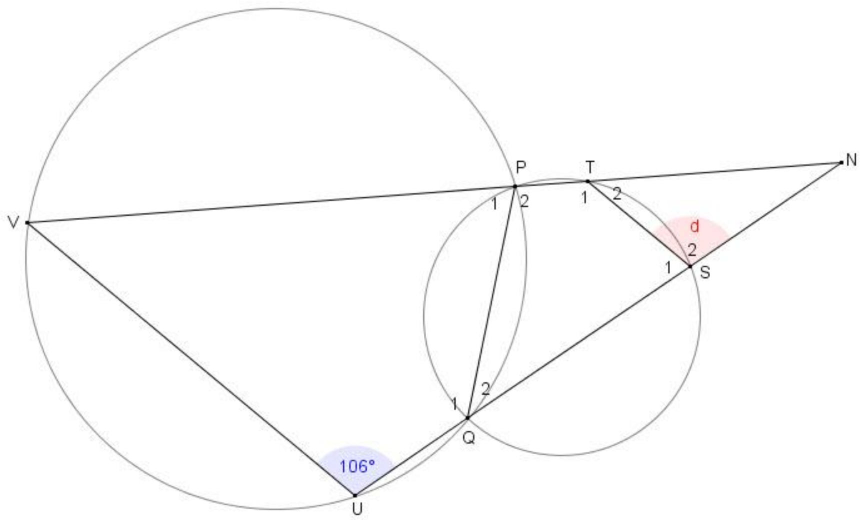
1. Calculate the size of a and b.



2. Calculate the size of c.



3. Calculate the size of d.



THEOREM 6: TANGENTS TO A CIRCLE

The first investigation is about a tangent and a radius of a circle. A *tangent* to a circle is a line that meets the circle at just one point and *perpendicular* means 90°. To solve problems and do proofs in this section you should also know the Theorem of Pythagoras.

Investigation 6-1: Drag the points; what do you notice about the angle between the tangent and radius? Describe this relationship in your own words (make a conjecture).

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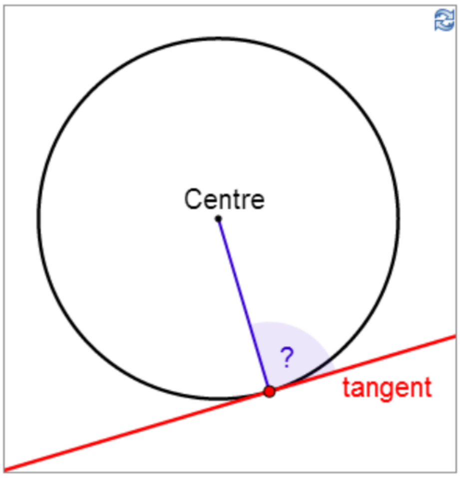

We will accept this conjecture and use it to prove other conjectures. This is why we use the word axiom. An axiom is a statement that we accept as true. You will notice that we also use all the results about congruency and triangles that we established in earlier grades as axioms to prove the circle conjectures.

Axiom: *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

An axiom is a statement that we accept as true without proving it. The next investigation is about the relationship between two tangents drawn from the same point outside the circle.

Investigation 6-2: Drag the points; what do you notice about the length of the tangents? Describe this relationship in your own words (make a conjecture).

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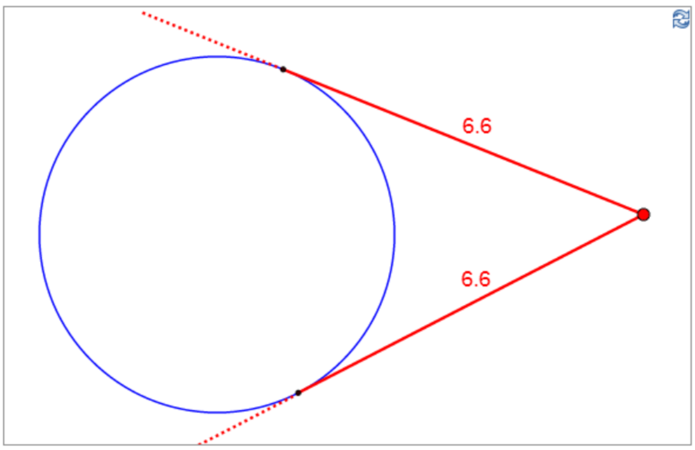
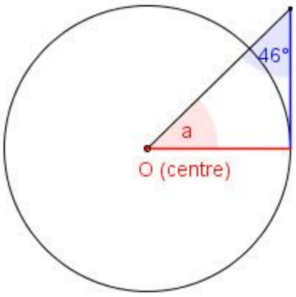
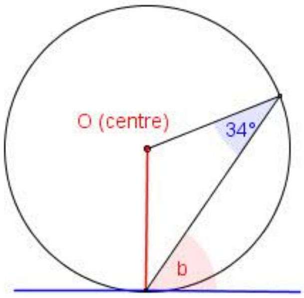


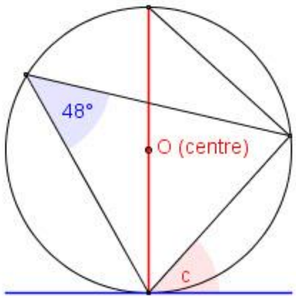
Conjecture 6: *If two tangents are drawn from the same point outside the circle, then they are equal in length.*

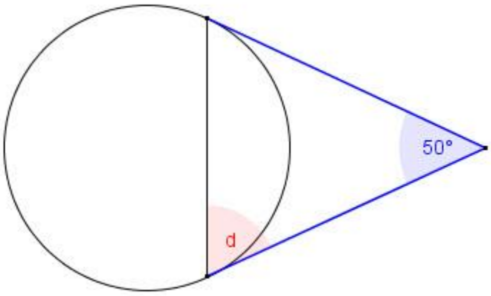
Exercise 6.1

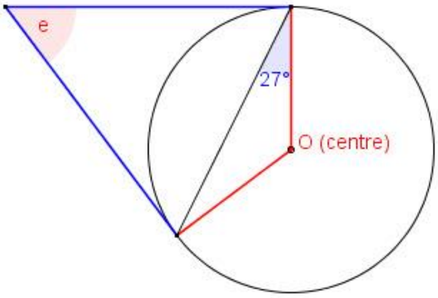
Calculate the size of *a, b, c, d* and *e*. The red line is a radius and the blue line is a tangent.











PROOF OF CONJECTURE 6

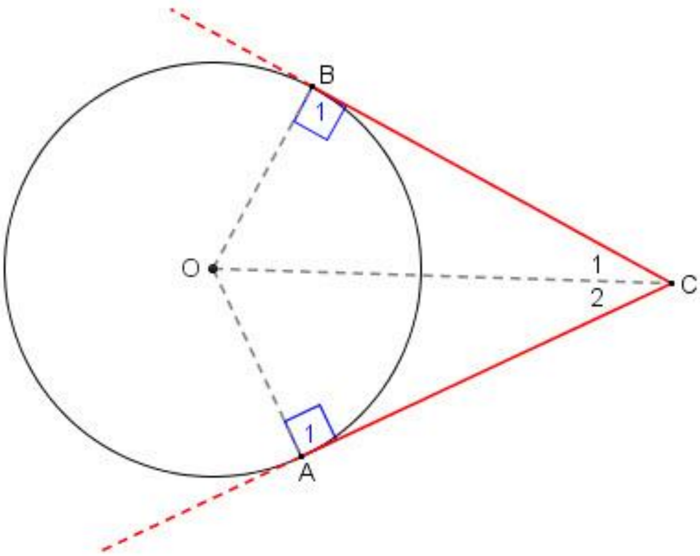
We will accept and use the following axiom to prove Conjecture 6:

Conjecture 6: *If two tangents are drawn from the same point outside the circle, then they are equal in length.*

An axiom is a statement that we accept as true. You will notice that we also use all the results about congruency and triangles that we established in earlier grades as axioms to prove the circle conjectures.

Axiom: *A tangent to a circle is perpendicular to the radius, drawn to the point of contact.*

Given: Circle O with tangents BC and AC from point C.
Construction: Draw OB, AO and OC.
Required to prove: BC = AC

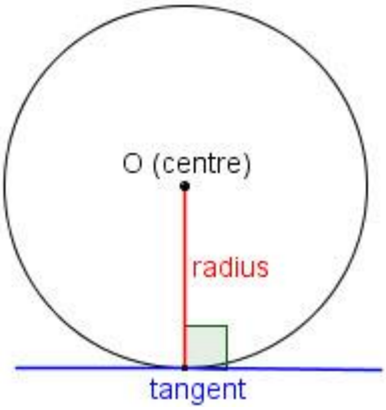


Proof:
In ΔOBC and ΔOAC:
OC is a common side
OB = OA (radii)
$\hat{B}_1 = \hat{A}_1$ (Axiom: a tangent is perpendicular to the radius)
Hence ΔOBC ≡ ΔOAC (90°angle, side, hypotenuse)
Therefore BC = AC

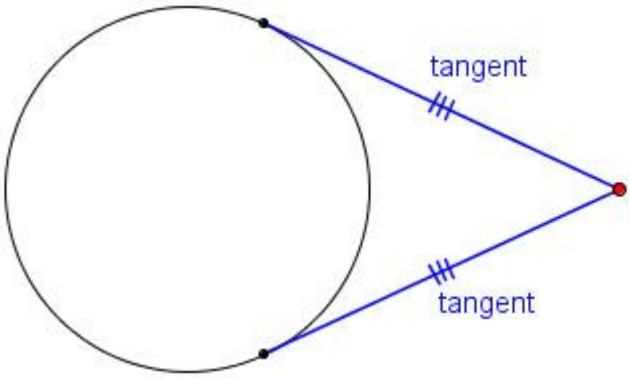
Theorem 6: If two tangents are drawn from the same point outside the circle, then they are equal in length.

REVIEW

Axiom: A tangent to a circle is perpendicular to the radius drawn to the point of contact.

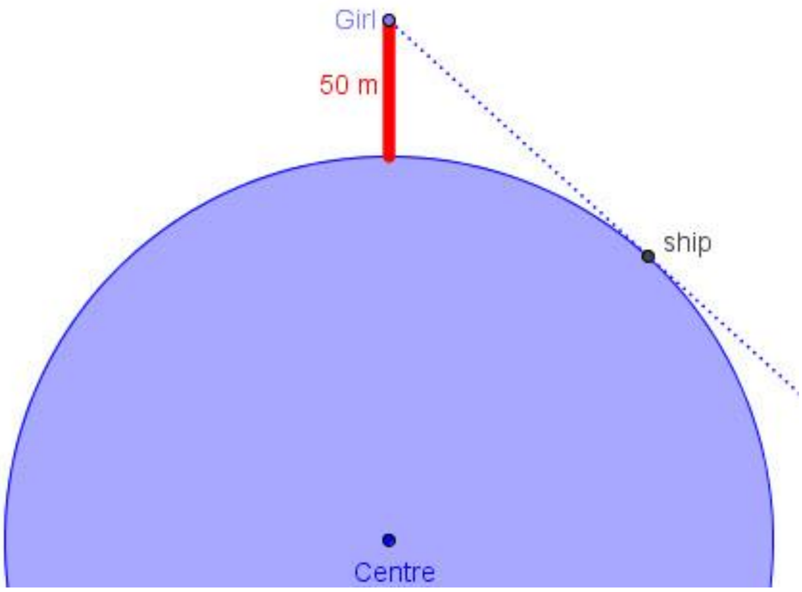


Conjecture 6: If two tangents are drawn from the same point outside the circle, then they are equal in length.

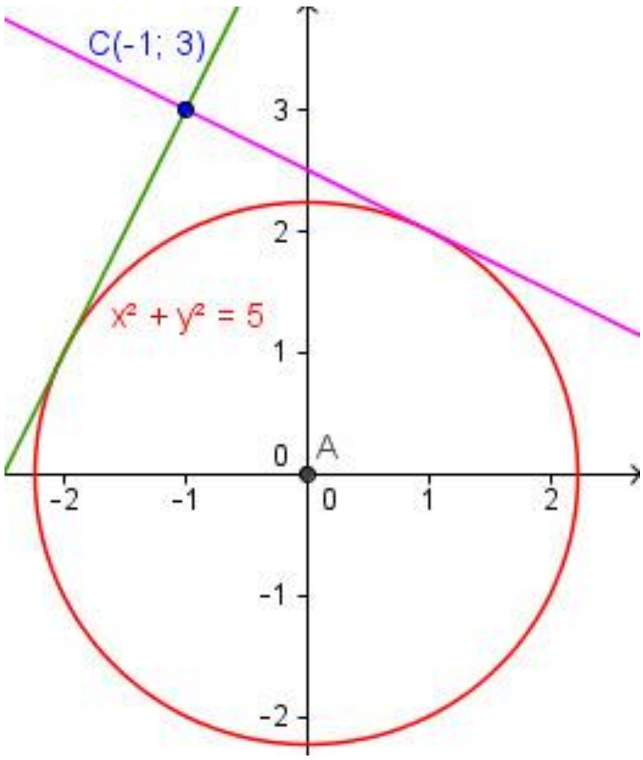


Exercise 6.2

1. A girl sits in a building, 50 m high, and stares at the sea looking for ships. How far can she see? Assume that the circumference of the earth is 42 650 km.



2. Determine the equations of the tangents to the circle $x^2 + y^2 = 5$ through point C(−1; 3).



APPLICATIONS OF THEOREM 6

Investigation 6-3: The circle is inscribed in a quadrilateral ABCD. Drag points A and C. Add the sum of the two opposite sides of the quadrilateral; what do you notice? Make a conjecture and prove the conjecture.

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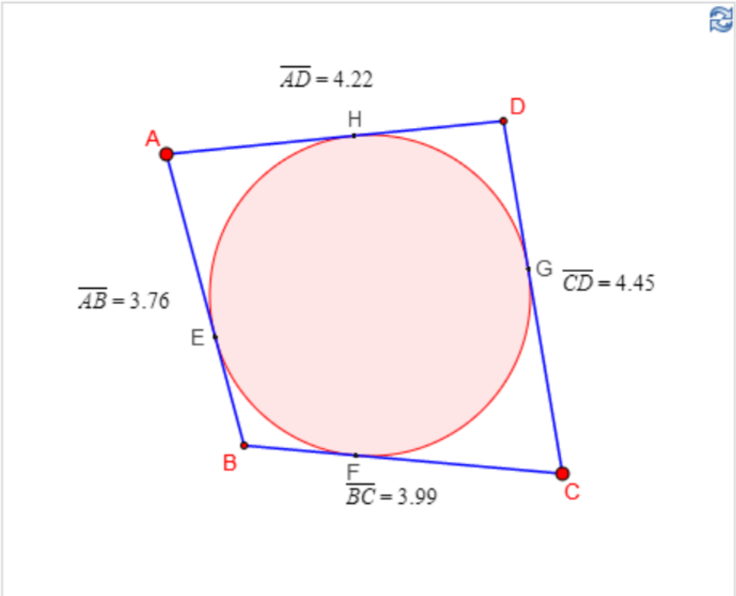
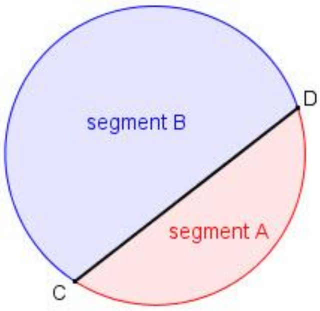


THEOREM 7: ANGLE BETWEEN A TANGENT AND A CHORD



In the diagram chord CD divides a circle in two areas or segments: segment A (red) and segment B (blue).
The *alternate* segment refers to the *other* segments. The alternate of segment A, is segment B.

This next investigation is about the relationship between the tangent to a circle and the chord drawn from the point of contact.

Investigation 7-1: Drag the points; what do you notice about the length of the tangents? Describe this relationship in your own words (make a conjecture).

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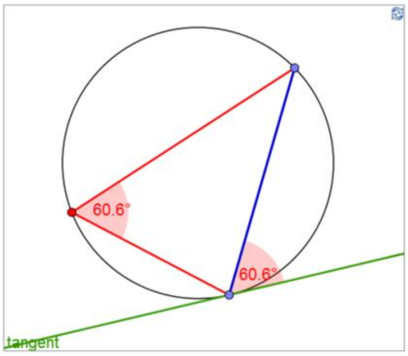

Conjecture 7a: *The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment.*

Why do you think it is necessary to add the requirement about the alternate segment to Conjecture 7a? You can use the next sketch to investigate the converse of the conjecture.

Investigation 7-2: Drag the vertices until the line is tangent to the circle; what do you notice about the angles?

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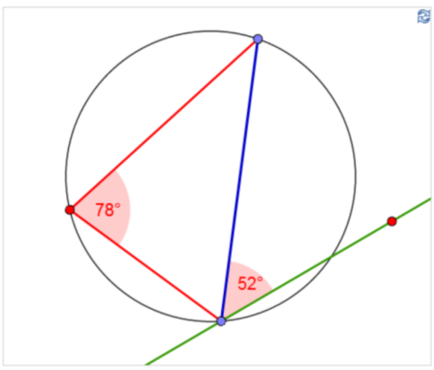
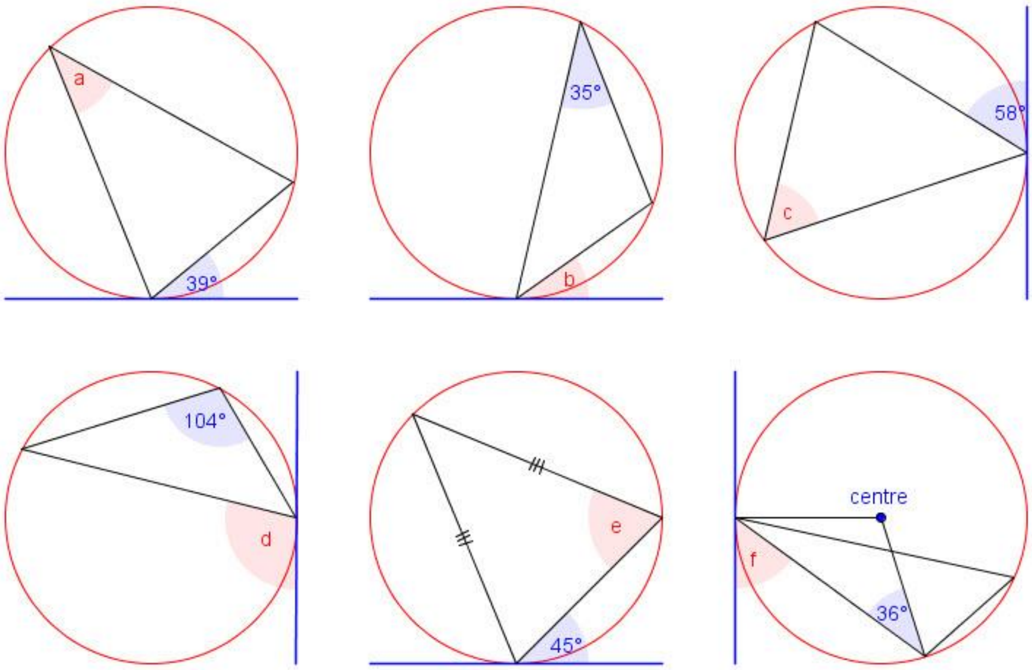


Conjecture 7b*: If a line is drawn through an endpoint of a chord, forming an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.*

Exercise 7.1

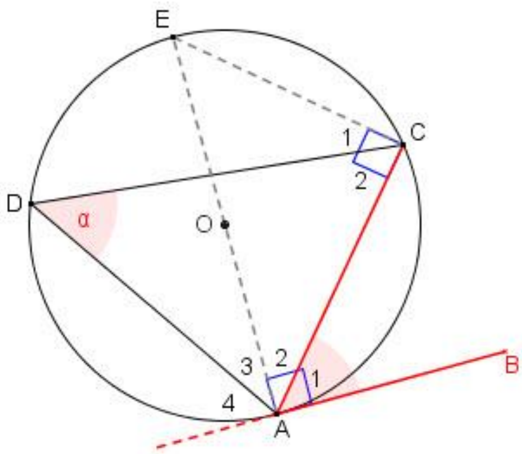
Calculate the size of *a, b, c, d, e* and *f*.



PROOF OF CONJECTURE 7

Conjecture 7a: *The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment.*

Given: Circle O with a tangent AB at A.
Construction: Draw diameter AOE and chord CE. Let $\widehat{D} = \alpha$.
Required to prove: $\hat{A}_1 = \alpha$ & $\hat{A}_4 = \hat{C}_2$



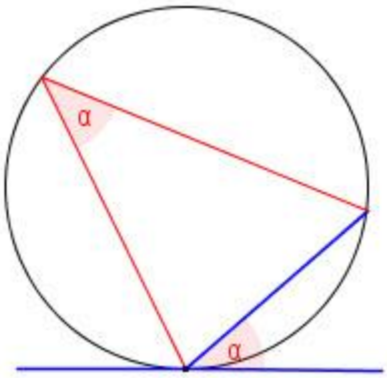
Proof
Draw diameter AE and join F and C.

$\widehat{D} = \hat{E} = \alpha$	(Angles on the circle subtended by chord AC)
and $\hat{C}_1 + \hat{C}_2 = 90°$	(Angles in a semi-circle)
then $\hat{A}_2 = 90° - \alpha$	(The sum of the interior angles of ΔAEC = 180°)
but $\hat{A}_1 + \hat{A}_2 = 90°$	(Assumption: a tangent is perpendicular to the radius)
$\hat{A}_1 + (90° - \alpha) = 90°$	($\hat{A}_2 = 90° - \alpha$)
$\hat{A}_1 = \alpha$	
$\hat{A}_4 = 180° - \hat{A}_2 - \hat{A}_3 - \alpha$	(Angles on a straight line)
and $\hat{C}_2 = 180° - \hat{A}_2 - \hat{A}_3 - \alpha$	(Sum of the interior angles of ΔACD)
therefore $\hat{A}_4 = \hat{C}_2$	

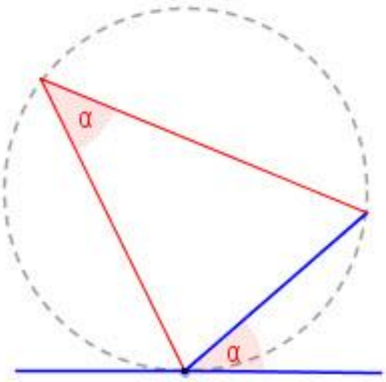
Theorem 7a: The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment.

REVIEW

Theorem 7a: The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.

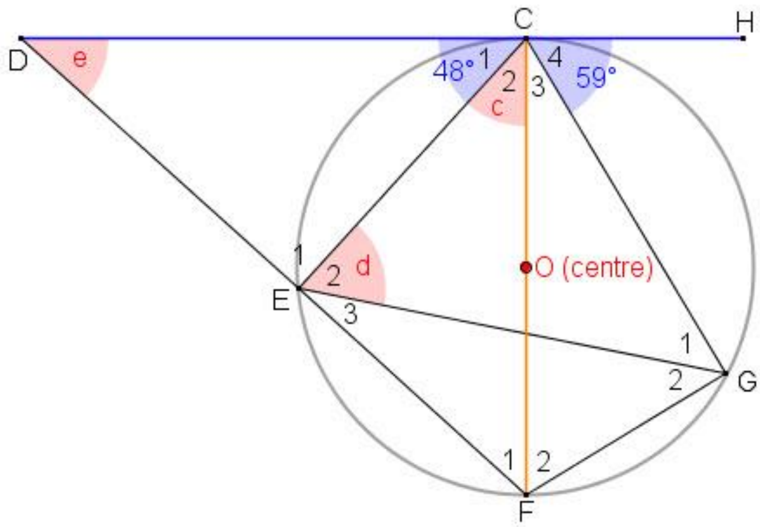


Theorem 7b: If a line is drawn through an endpoint of a chord, forming an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.

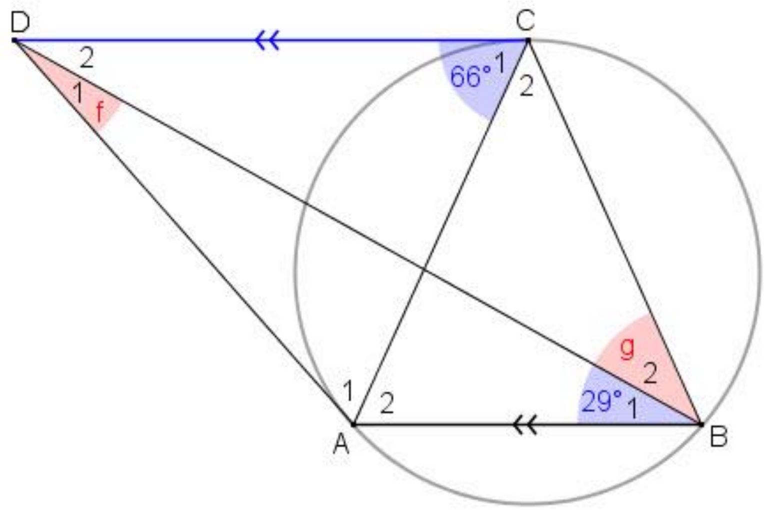


__Exercise 7.2__

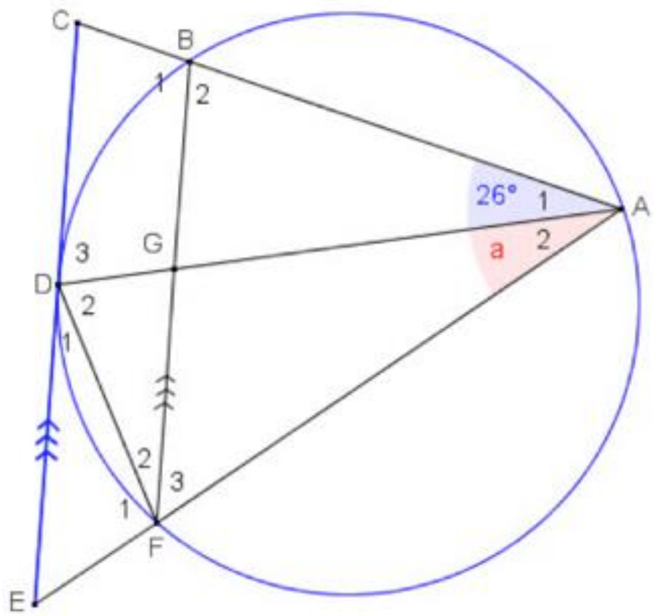
1. Calculate the size of angles *c*, *d* and *e* if *DH* is a tangent to the circle. O is the centre of the circle. Give a reason for each statement.



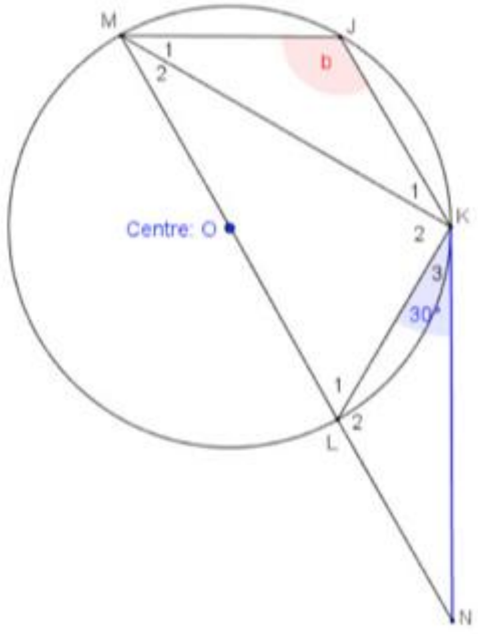
2. Calculate the size of angles *f* and *g* if *DC* is a tangent to the circle. $DC \parallel AB$. Give a reason for each statement.



3. Calculate the size of angle *a* if all the blue line segments are tangents to the circles. Give a reason for each statement.



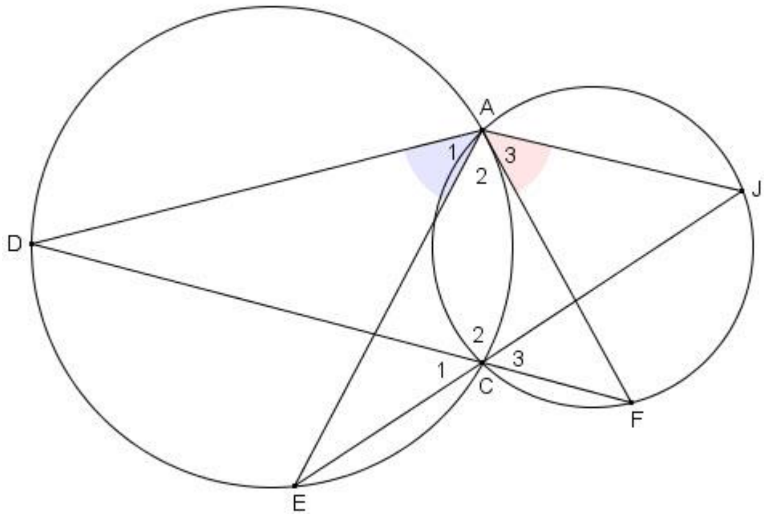
4. Calculate the size of angle *b* if all the blue line segments are tangents to the circles. Give a reason for each statement.



COMPLEX PROBLEMS (RIDERS)

In this section you will use the circle geometry theorems to solve more complex problems. Sometimes they refer to a complex problem as a rider. In all the problems to date we have used the theorems to calculate angles. In this section we will use the theorems to prove angles in more complex problems equal. These relationships are not only valid for a specific number but is for any angles. Let's look at an example:

Prove that $\hat{A}_1 = \hat{A}_3$.



Proof

$\hat{A}_1 = \hat{C}_1$ (Theorem 4a: Angles subtended by a chord at the circle are equal)

but $\hat{C}_1 = \hat{C}_3$ (Vertically opposite angles are equal)

and $\hat{C}_3 = \hat{A}_3$ (Theorem 4a: Angles subtended by a chord at the circle are equal)

Hence $\hat{A}_1 = \hat{A}_3$

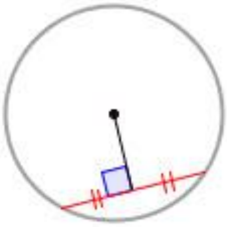
In summary the chain of reasoning is: $\hat{A}_1 = \hat{C}_1 = \hat{C}_3 = \hat{A}_3$.

But before we look at more complex problems let's summarise what you have learnt.

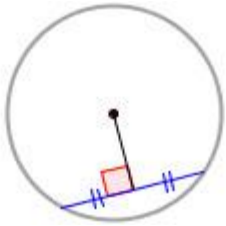
SUMMARY OF CIRCLE GEOMETRY THEOREMS

In this section you will use the circle geometry theorems to solve more complex problems. In the following summary the given "if" information is in blue and the conclusion "then" in red:

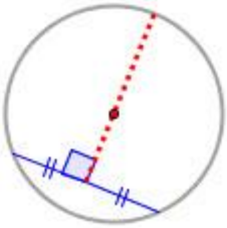
Theorem 1a: The line drawn from the centre of a circle perpendicular to a chord bisects the chord.



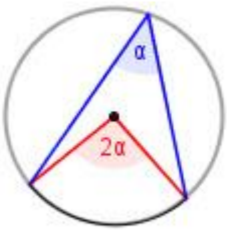
Theorem 1b: If a line is drawn from the centre of a circle to the midpoint of a chord, then that line is perpendicular to the chord.



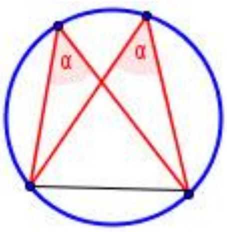
Theorem 2: The perpendicular bisector of a chord passes through the centre of the circle.



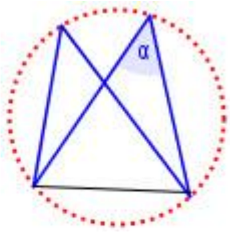
Theorem 3: The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre).



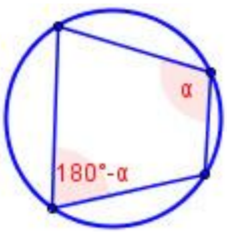
Theorem 4a: Angles subtended by a chord at the circle on the same side of the chord are equal.



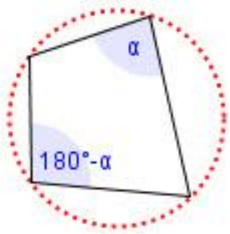
Theorem 4b: If a line segment joining two points subtends equal angles at two other points on the same side of a line segment, these four points are concyclic.



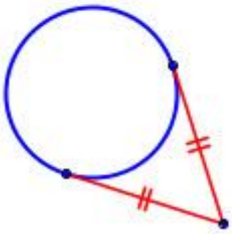
Theorem 5a: The opposite angles of a cyclic quadrilateral are supplementary.



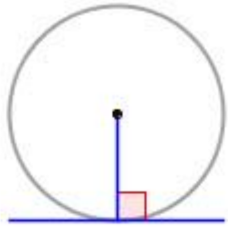
Theorem 5b: If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.



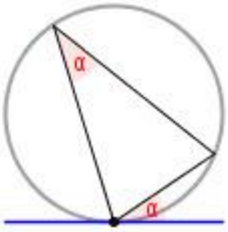
Theorem 6: Two tangents drawn to a circle from the same point outside the circle are equal in length.



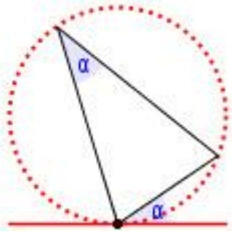
Axiom: A tangent to a circle is perpendicular to the radius drawn to the point of contact.



Theorem 7a: The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment.



Theorem 7b: The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.



Exercise 8.1

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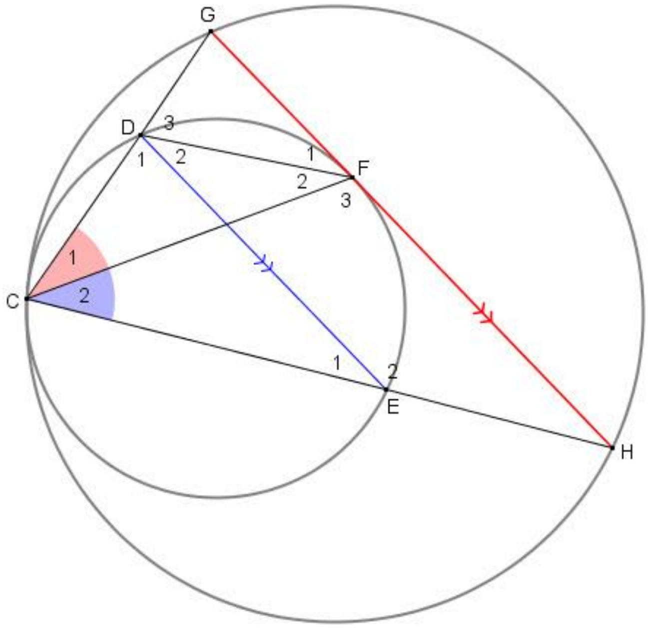
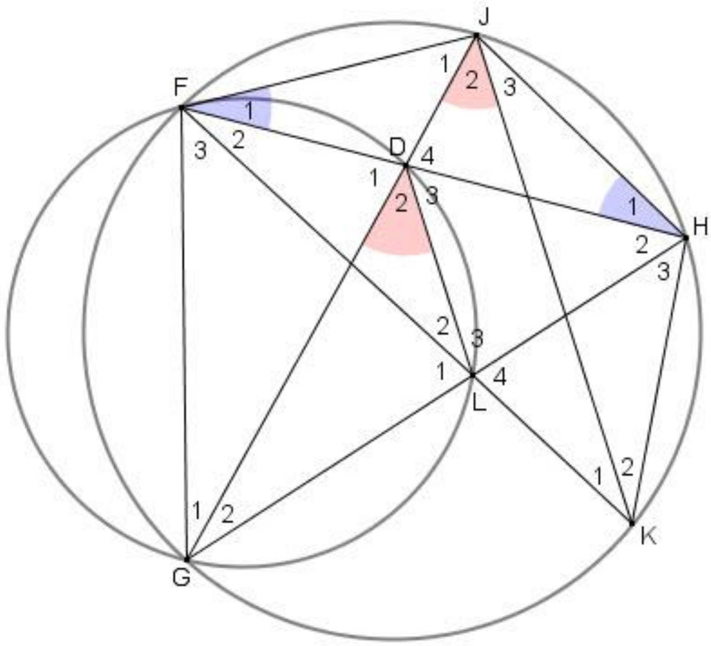
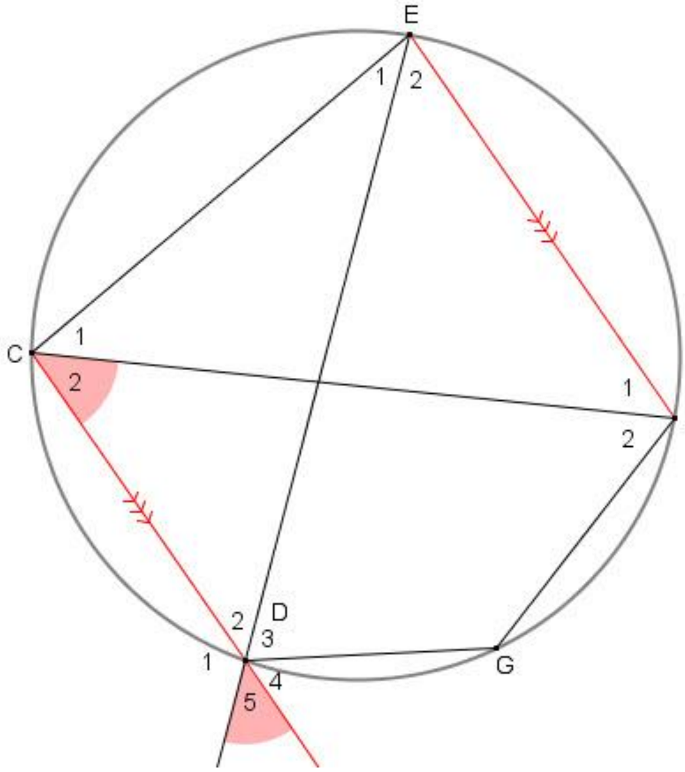
1. If $GH \parallel DE$ and GH is a tangent to the circle, prove that $\hat{C}_1 = \hat{C}_2$.



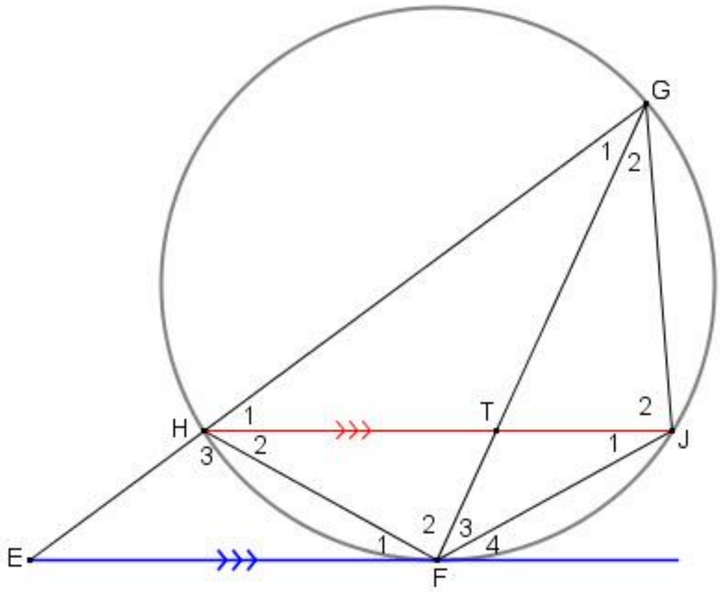
2. *FJ* is a tangent to circle *FGLD*.
   Prove that
   a) $\widehat{D}_2 = \hat{J}_2$
   b) ΔFJH is isosceles.



3. If *EF* ∥ *CD*, prove that $\widehat{D}_5 = \hat{C}_2$.



4. If *EF* is a tangent to the circle and *EF* ∥ *HJ* prove that $\widehat{G}_1 = \widehat{G}_2$.



HOW DO I PROVE A QUAD IS CYCLIC OR A LINE IS A TANGENT?

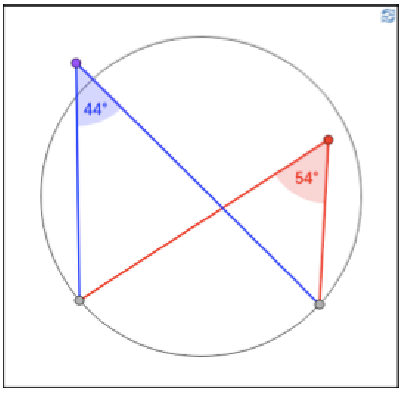
As seen in the previous questions we can be asked directly or indirectly to prove that two angles are equal. For example, instead of asking you to prove that $\hat{F}_1 = \hat{H}_1$, I can also ask you to prove that ΔFJH is isosceles. In this case you first need to interpret the question in terms angles.

Prove that two lines are parallel
Prove a pair of corresponding angles or alternate angles equal or that the sum of a pair of co-interior angles is supplementary (180°).

Prove that a quadrilateral is cyclic (or that four points are concyclic)
We can use Theorems 4b or 5b to prove that a quadrilateral is cyclic. Cyclic means that it is possible to construct a circle through all four vertices of the quadrilateral.

Investigation 8-1: Drag the point onto the circle. What do you notice about the angles?

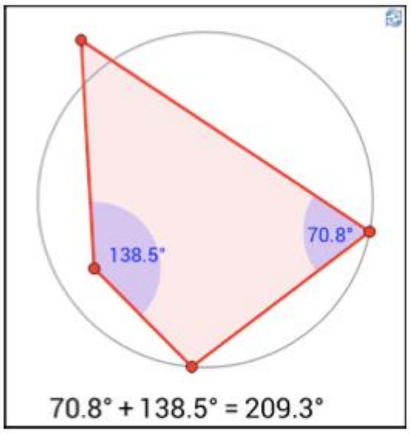
Click here for the Interactive



Theorem 4b: If a line segment joining two points subtends equal angles at two other points on the same side of a line segment, these four points are concyclic.

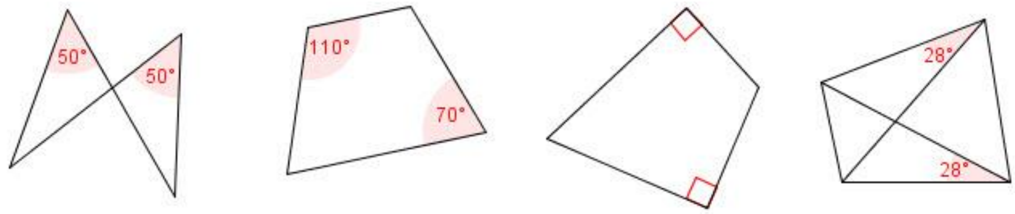
Investigation 8-2: Drag the point onto the circle. What do you notice about the angles?

Click here for the Interactive

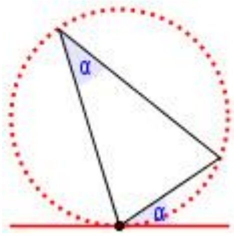


Theorem 5b: If two opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

Examples of cyclic quadrilaterals:

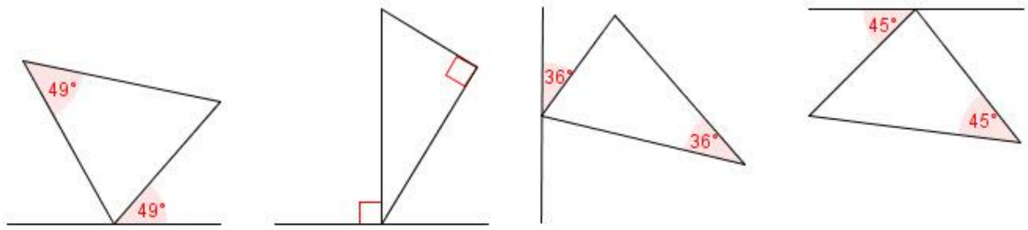


Proof that a line is a tangent to a circle
We can use the Theorems 4b or 5b to prove that a quadrilateral is cyclic (that it is possible to construct a circle through all four vertices).



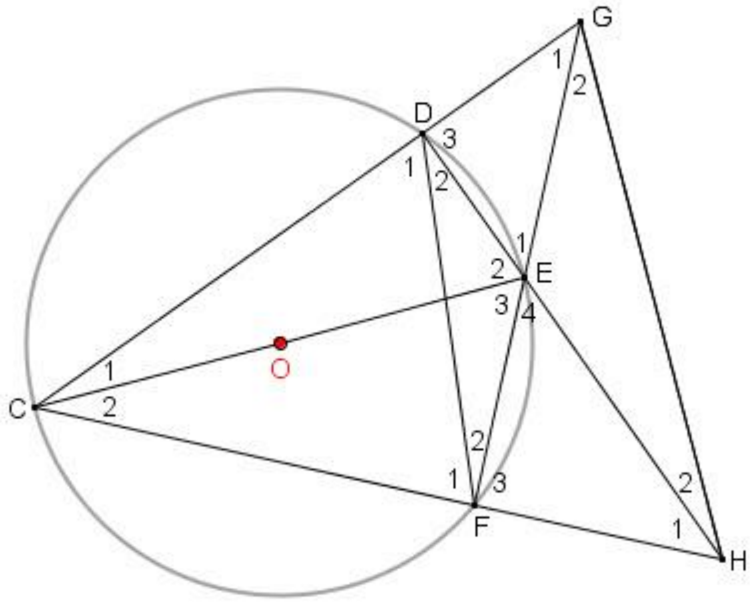
Theorem 7b: If a line segment joining two points subtends equal angles at two other points on the same side of a line segment, these four points are concyclic.

Examples of tangents to a circle through three points

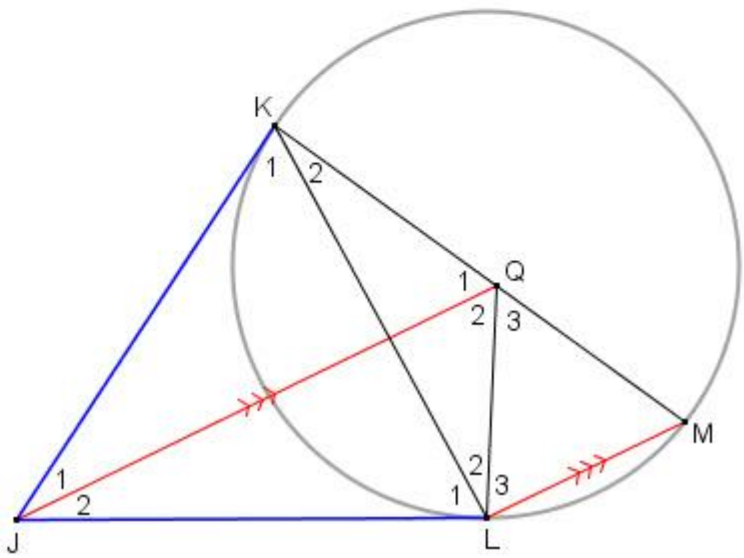


Exercise 8.2

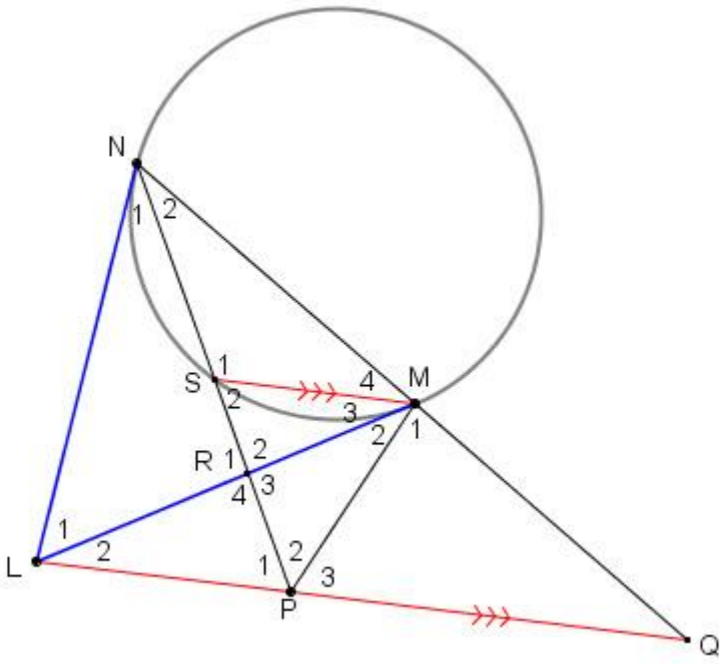
1. If O is the centre of the circle, prove that $DGHF$ is a cyclic quadrilateral.



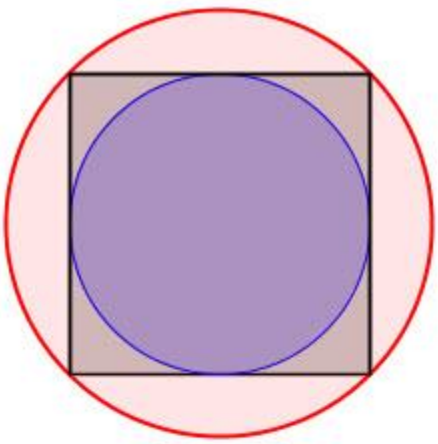
2. If JL and JK are tangents to the circle and $JQ \parallel LM$, prove that $JKQL$ is a cyclic quadrilateral.



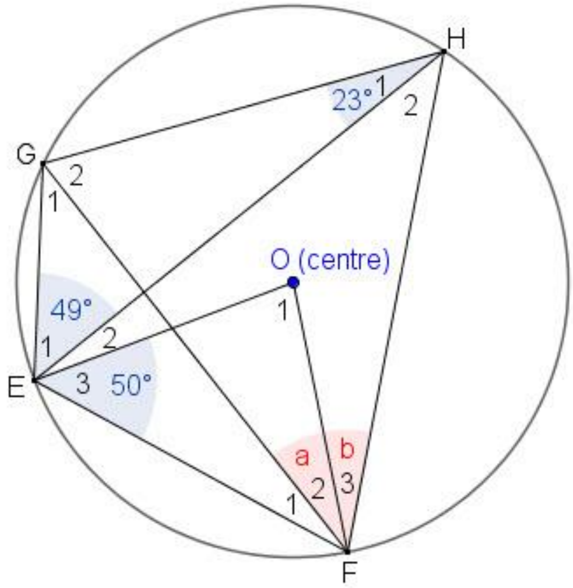
3. If *LN* and *LM* are tangents to the circle and *SM* ∥ *LQ*, prove that *LPMN* is a cyclic quadrilateral.



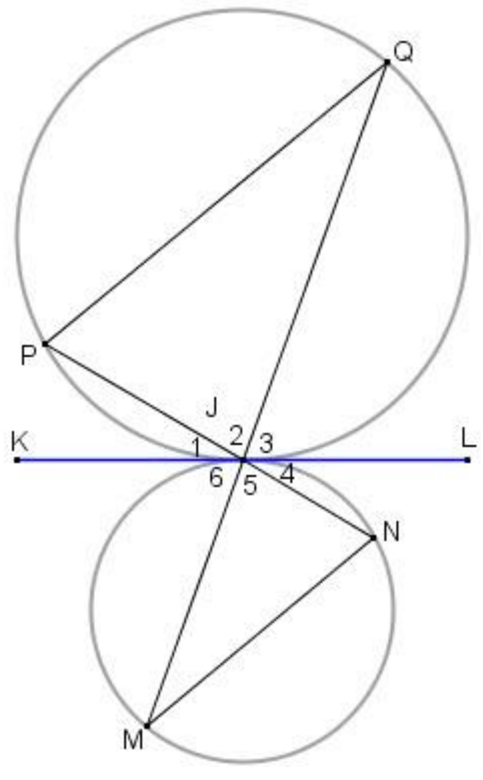
4. If the area of the square is 16 cm², calculate the area between the two circles.



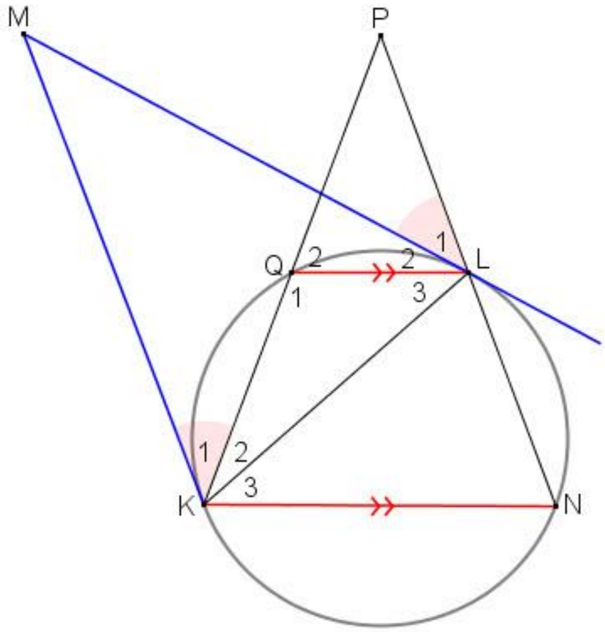
5. Calculate the sizes of *a* and *b*.



6. Prove that *PQ* || *MN* if *KL* is a common tangent.



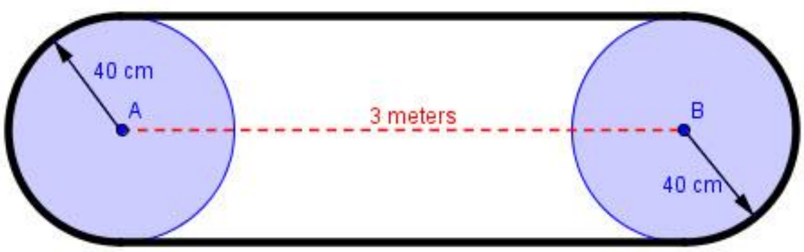
7. If *MK* and *ML* are tangents to the circle and $KN \parallel QL$, prove that *MKLP* is a cyclic quadrilateral.



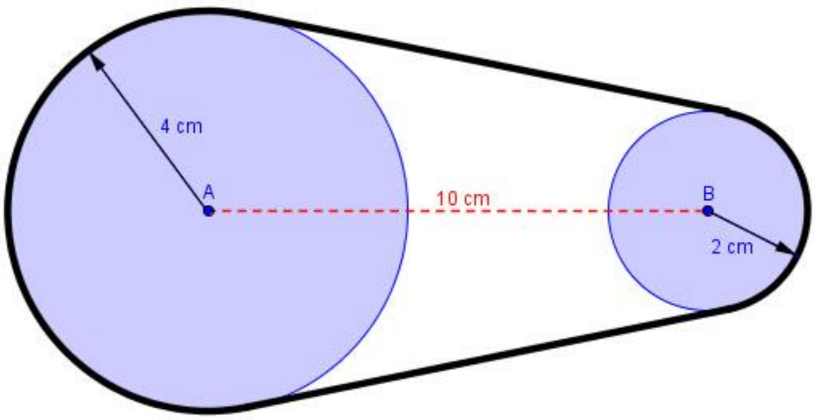
PROJECT

1. All-terrain tracks help propel a machine over all types of terrain. The question is about the length of track of an excavator.

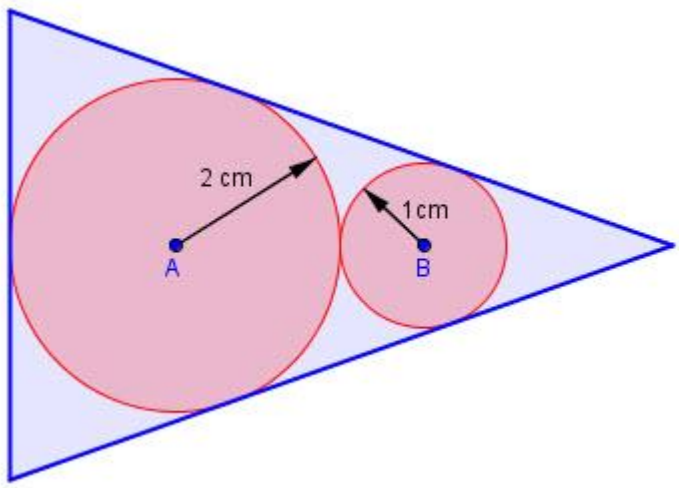
The radius of both wheels is 40 cm and the distance between the wheels is 3 metres. Calculate the total length of the track.



2. A bicycle chain transfers power from the pedals to the wheels of a bicycle. The chain of your bicycle connects two gears with centres A and B. The radii of the gears are 2 cm and 4 cm, and the distance between their centres is 10 cm. Find the total length of chain needed to connect the two gears.



3. Calculate the area of the blue shaded area inside the triangle.



AXIOMATIC SYSTEM

The ancient Greek mathematician Euclid of Alexandria lived about 300 BC. He is known as the father of geometry. He realised the importance of assumptions (axioms or postulates) and reorganised the known geometry at that stage into an axiomatic system. Euclid wrote a series of 13 books, which is known as the Elements. The high school geometry we all know are just small selections from his book. This is why the geometry in this book is known as Euclidean Geometry.

Geometry uses the terms as triangle, line, circle, quadrilateral and so on. But how do you explain these concepts to somebody else? You can explain, for example, that a *triangle* consists of three *line segments* that connect three *non-collinear points*. But then you have to explain the terms *line segments* and *non-collinear points*. Any attempt to explain the meaning of every word would lead to an infinite cycles of regression. You have to start somewhere. We will call these starting points *undefined terms*. The geometry curriculum uses the following undefined terms: point, line, etc. Once you have agreed on the undefined terms, you can use them to explain other concepts (definitions), for example: a triangle consists of three segments connecting three non-collinear points.

Undefined terms: point, line, plane, between, segment.

Definitions

- A *line segment* consists of two points plus all the points between them on the line containing these two points.
- A *perpendicular bisector* of a line segment is the line which is perpendicular to the line segment and passes through the midpoint of the line segment.
- Points are *collinear* if they lie on the same line.
- A *triangle* consists of three segments connecting three points that are not on one line.
- A *circle* is the set of all points in a plane at a fixed distance, called the radius, from a fixed point, called the centre.
- The segment bound by the circle is called a *chord*. The area of a circle cut off by a chord is called a *circle segment*.
- A line touching the circle in one place is called a *tangent*.
- An *angle* is formed by two segments with a common vertex.
- If two sides of a triangle have the same measure it is an *isosceles triangle*.

AXIOMS

A mathematician wants to understand a relationship and wants to ascertain whether the discovery or relationship always exists. We need to justify our claims rigorously. The process of proving is known as deductive reasoning. Deductive reasoning is a method that uses logic to make inferences from relationships we regard as true. We use deductive reasoning in mathematical proofs and call the conjectures that we prove through deductive reasoning, theorems. We use theorems and previous results to prove new theorems. This makes proofs messy because it is never clear what results can be accepted and used in order to prove a theorem. Just as the vocabulary of an axiomatic system has to contain some fundamental undefined terms, so some statements have to be accepted as basic without proof. Euclid and other mathematicians realised that we do not have any choice but to accept certain 'basic truths' (Euclidian geometry is named after Euclid). We call these truths axioms or postulates. Axioms are therefore unproved assumptions that we make about undefined terms. It is important to prove any so-called 'facts' if we want to use them in a proof. In geometry we use congruency and previously learnt relationships to prove our conjectures. Let's analyse the structure of the proofs of the Theorems 1 to 7.

The table shows the reasons that we used in each of the proofs of the different theorems:

Theorem	Reasons used in the proof of the theorem
1	Conditions for congruency of triangles.
2	Conditions for congruency of triangles. The sum of the interior angles of a triangle is 180°.
3	The measures of the two base angles of an isosceles triangle are equal. Exterior angle of a triangle is equal to the sum of interior opposite angles
4	Theorem 3
5	Theorem 3 The sum of the angles around a point is 360°.
6	Conditions for congruency of triangles. A tangent is perpendicular to the radius drawn at the point of contact with the circle.
7	Theorem 3 The sum of angles on a straight line is 180°. If two angles are a linear pair, the sum of their measure is 180°. A tangent is perpendicular to the radius drawn at the point of contact with the circle.

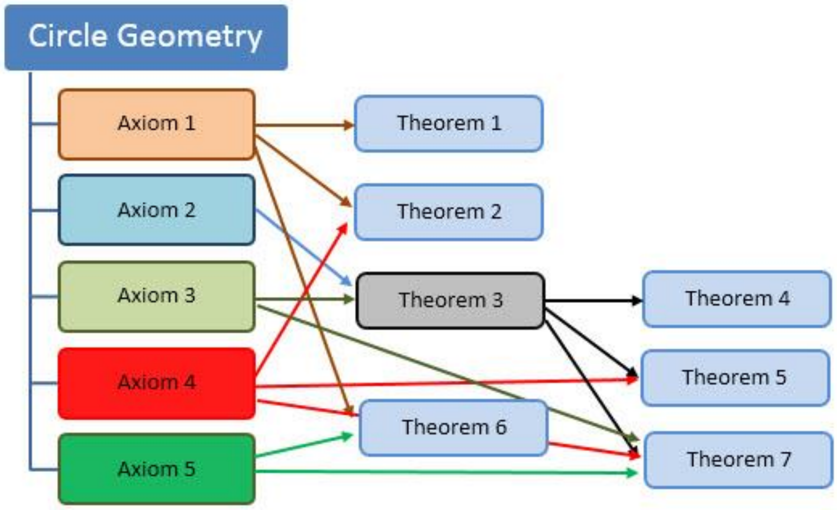
We will refer to these reasons as *axioms* (results that were established earlier). We assume these results without an attempt to prove them. This is our starting point. We can group some of the related reasons together because they are all based on the same principle:

- Exterior angle of a triangle is equal to the sum of interior opposite angles.
- The sum of the interior angles of a triangle is 180°.
- The sum of the angles around a point is 360°.

We can therefore list our axioms:

- Axiom 1: Conditions for congruency of triangles.
- Axiom 2: The measures of the two base angles of an isosceles triangle are equal.
- Axiom 3: Exterior angle of a triangle is equal to the sum of interior opposite angles.
- Axiom 4: If two angles are a linear pair, the sum of their measure is 180°.
- Axiom 5: A tangent is perpendicular to the radius, drawn at the point of contact with the circle.

Axiomatic geometry starts with an analysis of the proofs in order to determine the axioms. You will notice from this structure the importance of Theorem 3 in the proofs. The following figure is a representation of the structure of the circle geometry in this book:



LOGIC AND PROOFS

One of the most common forms of deductive logic was developed by Aristotle (384-322 BC). It is commonly known as Modus ponens. The first premise is the “if-then” or “P implies Q” statement. The second premise is that P, the conditional claim, is true. From these two premises it can be logically concluded that Q must be true as well. Modus ponens is therefore a valid argument form of type: If “P implies Q” is true and if P is true then Q is true. An argument using modus ponens is said to be deductive. An inductive argument is one in which the conclusion goes beyond the premises.

Exercise 9.1

---

1. Prove Theorem 1b: If a line is drawn from the centre of a circle to the midpoint of a chord, then that line is perpendicular to the chord.
2. Prove Theorem 4b: If a line segment joining two points subtends equal angles at two other points on the same side of a line segment, these four points are concyclic.
3. Prove Theorem 5b: If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.
4. Prove Theorem 7b: The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.

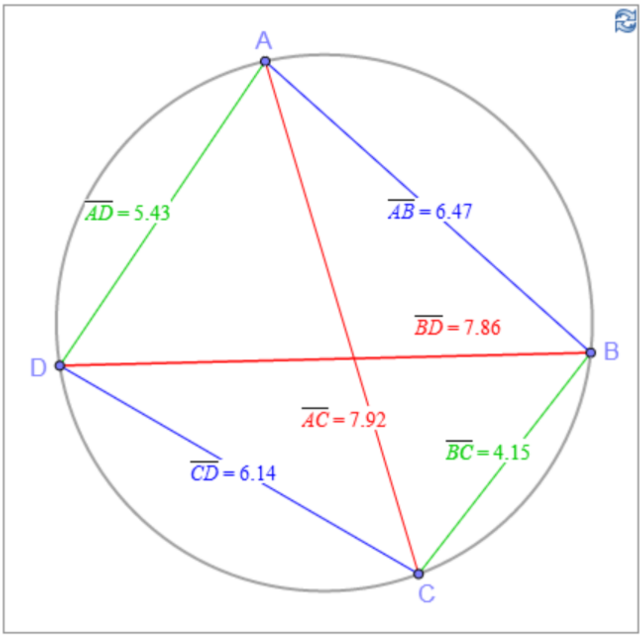
MORE CIRCLE RELATIONSHIPS

There are interesting circle properties that we do not discuss in our school curriculum, for example Ptolemy's theorem. This is one of the great theorems of geometry from the time of the ancient Greeks. Ptolemy or Claudius Ptolemæus lived around 90-170 AD in Alexandria and his great work, called Al Magest, covers almost all the astronomical knowledge of the ancient world.

PTOLEMY'S THEOREM

The theorem says that if A, B, C and D are points in order around a circle, then (AB)(CD) + (BC)(DA) = (AC)(BD). It is possible to prove these theorems by using grade 12 theorems. This is one of the great theorems of geometry from the time of the ancient Greeks; Ptolemy used this theorem in about 150 AD to calculate the lengths of chords in a circle.

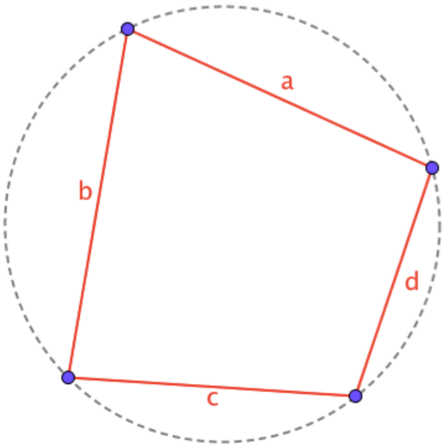
Ptolemy's theorem: The theorem says that if A, B, C and D are points in order around a circle, then (AB)(CD) + (BC)(DA) = (AC)(BD).



In other words Ptolemy's Theorem states that in a cyclic quadrilateral the sum of the products of the two pairs of opposite sides equals the product of its two diagonals.

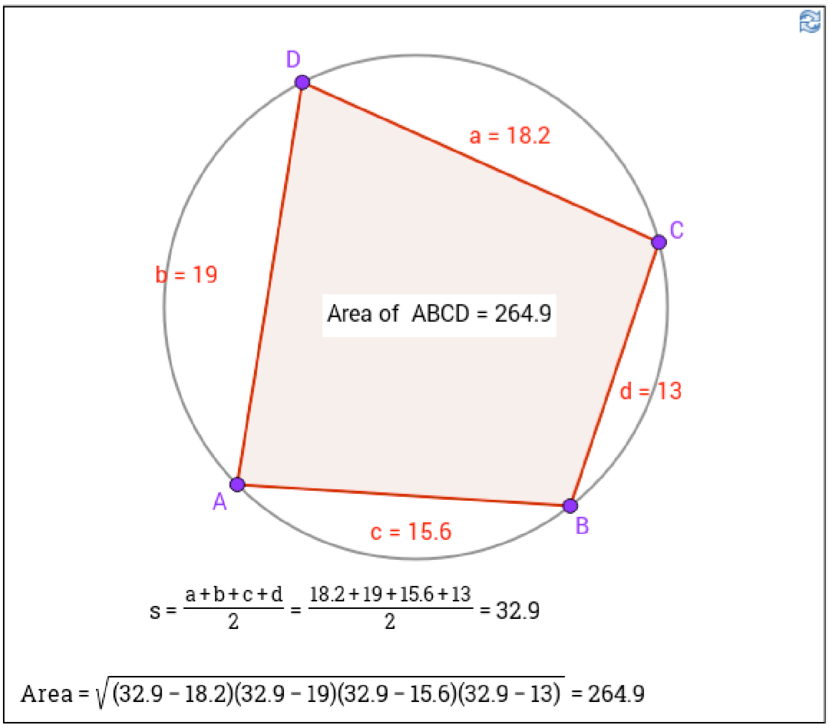
THE AREA OF A CYCLIC QUADRILATERAL

Brahmagupta was an Indian mathematician who lived in the 7th century. He develops a simple way to calculate the area of a cyclic quadrilateral if you know all the sides. If the sides' lengths are for example a, b, c and d then let s be half of the circumference: $s = \frac{a+b+c+d}{2}$



Investigation 10-1: Drag the vertices of the quadrilateral and use Brahmagupta's area formula to calculate the area.

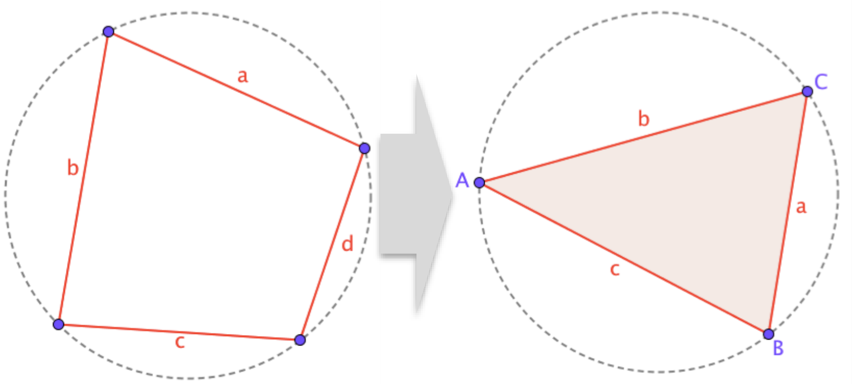
[Click here for the Interactive]



Brahmagupta conjecture: The area of the cyclic quadrilateral is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where $s = \frac{a+b+c+d}{2}$.

THE AREA OF A TRIANGLE

If we make one of the sides of the cyclic quadrilateral zero we get a triangle (all triangles are cyclic - it is always possible to construct a circumscribed triangle to a circle). Heron of Alexandria was an ancient Greek mathematician who lived about 10-70 AD. We can use Brahmagupta's conjecture for a cyclic quadrilateral and move D on top of A, therefore AD = d = 0.



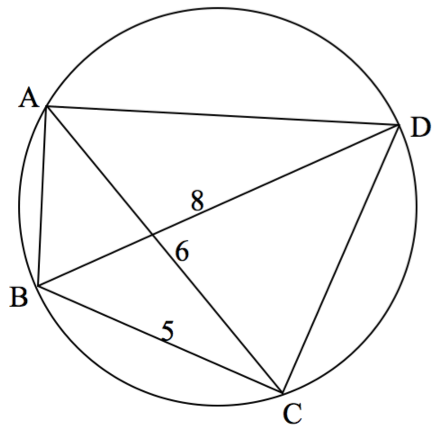
From the sketch it is clear that Heron's formula for the area of a triangle is just a special case of Brahmagupta's formula for the area of a cyclic quadrilateral. Let s be half of the circumference: $s = \frac{a+b+c+0}{2}$. The area of the cyclic quadrilateral is then: Area = (s − a) (s − b) (s − c) (s − 0). This can be simplified to a formula known as Heron's formula for the area of a triangle.

Heron's formula: The area of the triangle is $\sqrt{(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$.

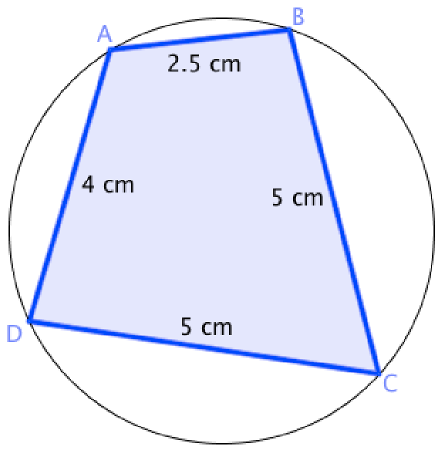
Exercise 10.1

1. Calculate the area of a triangle with sides 2 cm, 3 cm, and 4 cm.

2. Find the length AB in the figure below. Here, the circle has diameter 8 cm and BD is a diameter, length BC = 5 cm, and length AC = 6 cm.



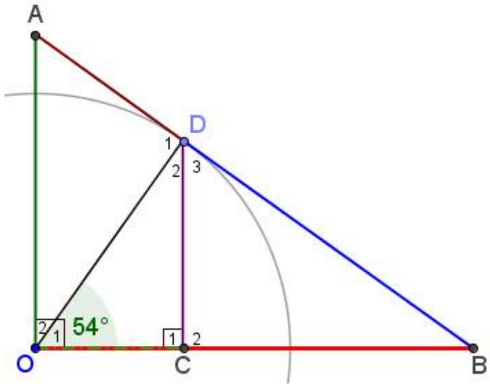
3. Calculate the area of quadrilateral ABCD.



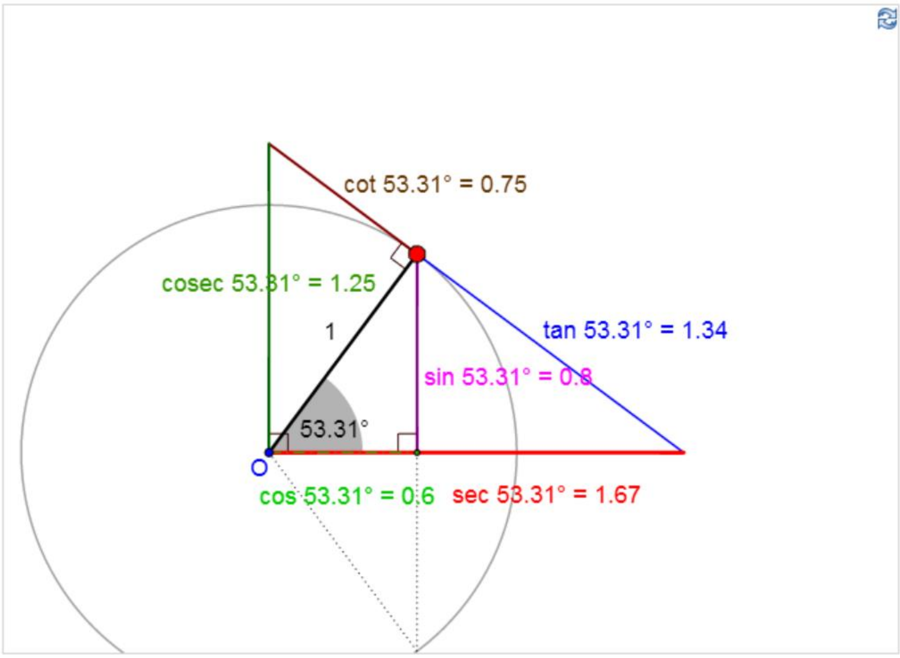
4. Prove the theorem of Pythagoras using Ptolemy's Theorem.

5. Prove the compound angle sine and cosine rule using Ptolemy's Theorem.

4. Calculate the sizes of all the angles if *AB* is tangent to the circle with centre *O* and radius *OD*. $AO \perp OB$ and $DC \perp OB$.



5. It was not easy to find the value of the trigonometric functions for each angle before calculators. The mathematician François Viète (1540-1603) used a construction to find these values. In the diagram the length of each segment gives the value of all six trigonometric functions if the radius OD has a length of 1 unit. Drag the red dot to find the values for an angle size:



Explain why this result works.

ANSWERS

Exercise 1.1
1. p = 40 mm
2. q = 50 mm

Exercise 1.2
1. 1 cm
2. 32,5 cm
3. proof
4. EG = 12 cm
5. 2,24 units

Exercise 2.1
1. (6; 4)

Exercise 3.1

a = 49°
b = 96°
c = 45°
d = 234°
e = 90°
f = 24°

Exercise 3.2
1. a = 52°
2. b = 38°
3. c = 54°
4. d = 32°
5. proof
6. Construct any circle and its diameter. Any diameter of a circle always subtends a right angle to any point on the circle.

Exercise 4.1

a = 52°
b = 32°
c = 33°
d = 102°
e = 24°
f = 40.5°
g = 40.5°

Exercise 4.2
1. a = 48°
2. b = 51°
3. c = 50°
4. d = 110°

Exercise 5.1

a = 96°
b = 63°
c = 115°
d = 63°

Exercise 5.2

1. $a = 41°$; $b = 64°$
2. c = 46°
3. d = 106°

Exercise 6.1

a = 44°
b = 56°
c = 48°
d = 65°
e = 54°

Exercise 6.2

1. radius = 6 788 km & distance = 26 km
2. Determine the equations of the tangents to the circle $x^2 + y^2 = 5$ through point C(−1; 3).

Exercise 7.1

$a = 39°$
$b = 35°$
$c = 58°$
$d = 104°$
$e = 67.5°$
$f = 54°$

Exercise 7.2

1. c = 42°; d = 59°; e = 42°
2. $f = 19°$; $g = 37°$
3. $a = 26°$
4. $b = 120°$

Exercise 8.1

1. $\hat{C}_1 = \hat{F}_1 = \hat{D}_2 = \hat{C}_2$
2. a) $\hat{D}_2 = \hat{F}_3 = \hat{J}_2$
   b) $\hat{F}_1 = \hat{G}_1 = \hat{H}_1$
3. $\hat{D}_5 = \hat{D}_2 = \hat{E}_2 = \hat{C}_2$
4. $\hat{G}_1 = \hat{F}_1 = \hat{H}_2 = \hat{G}_2$

Exercise 8.2

1. $\hat{D}_1 + \hat{D}_2 = 90°$ & $\hat{F}_1 + \hat{F}_2 = 90°$ therefore $\hat{D}_3 = \hat{F}_3 = 90°$
2. $\hat{L}_1 = \hat{M} = \hat{O}_1$
3. $\hat{N}_2 = \hat{M}_3 = \hat{L}_2$
4. 4π
5. $a = 27°$ & $b = 22°$
6. $\hat{Q} = \hat{J}_1 = \hat{M}$, therefore PQ || MN
7. $\hat{L}_1 = \hat{L}_5 = \hat{K}_3 = \hat{L}_3 = \hat{K}_1$

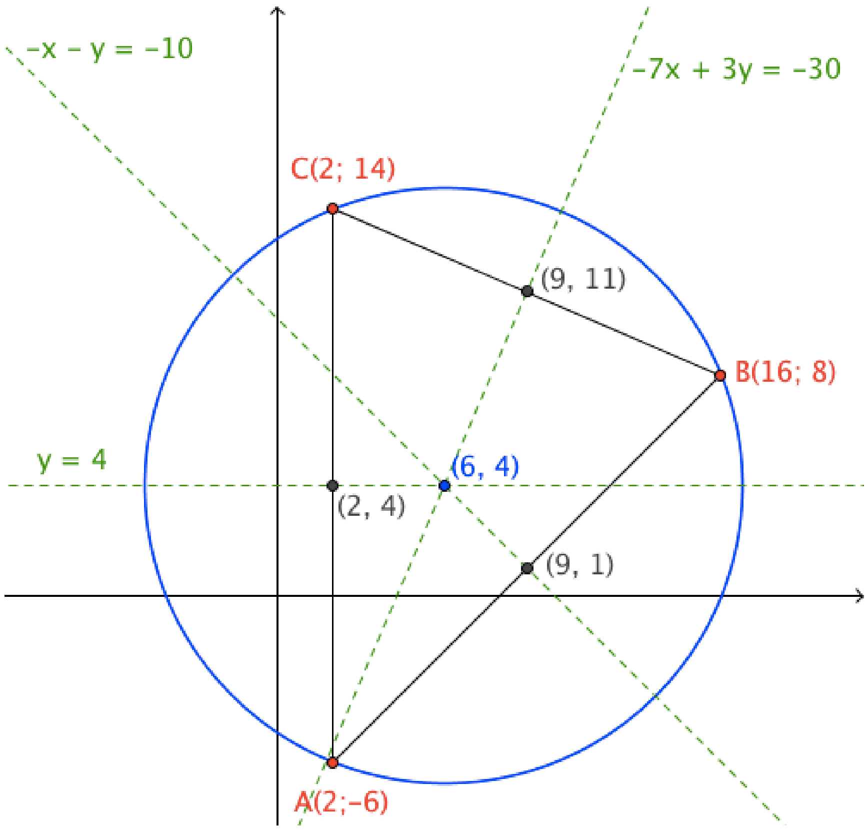
PROJECT

1. 8,51 m
2. 53,42 cm

SELECTED SOLUTIONS

Exercise 2.1

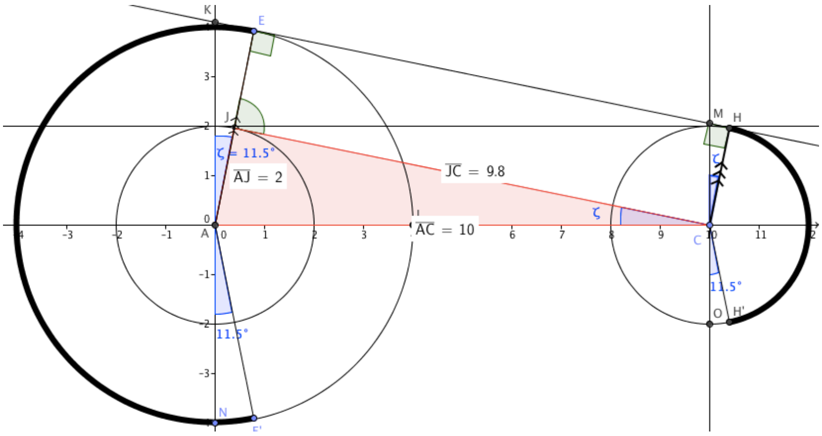
1. There is more than one way to find the centre of the circle. One way is to determine the equations of the two perpendicular bisectors and solve them simultaneously. Another way is to let $P(x; y)$ be the centre of the circle. Then $AP = BP$ and $AP = PC$ (radii). Let's try the first way that we mentioned. First, we are going to connect the points and then determine the equation of two of the perpendicular bisectors. Points $A(2; -6)$, $B(16; 8)$ and $C(2; 14)$ are on a circle.



The midpoint of segment AC is $(2; 4)$ and the equation of the perpendicular bisector of AC is $y = 4$. The midpoint of BC is $(9; 11)$ and the equation of the perpendicular bisector of BC is $-7x + 3y = -30$. The midpoint of AB is $(9; 1)$ and the equation of the perpendicular bisector of AB is $x + y = 10$. By solving the equations of any two perpendicular bisectors simultaneously we find the centre of the circle. The centre is the point $(6; 4)$.

Project

2. $\angle ACJ = \angle KAE = \zeta$ and $\tan\zeta = \frac{2}{10} = 0{,}2$ therefore $\zeta = 11{,}5°$.



Big arc EE': Angle = 180° + 11,5° + 11,5° = 203°

Big arc length = $\frac{203}{360} \times (circumference) = \frac{203}{360} \times (16\pi) = 28{,}34$ cm

Small arc HH': Angle = 180° − 11,5° − 11,5° = 157°

Big arc length = $\frac{157}{360} \times (circumference) = \frac{157}{360} \times (4\pi) = 5{,}48$ cm

EH: $JC^2 = 10^2 - 2^2$ then $JC = 9{,}8\ cm$ (Pythagoras)

Total chain length = 28,34 + 5,48 + 2(9,8) = 53,42 cm

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