

Sketching the general quadratic equation using dynamic geometry software

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This paper explores a geometrical way to sketch graphs of the general quadratic in two variables with Geometer's Sketchpad. To do this, a geometric procedure as described by De Temple is used, bearing in mind that this general quadratic equation (1) represents all the possible conics (conics sections), and the fact that five points (no three of which are collinear) uniquely determine a conic.

1. Introduction

This paper explores a geometrical way to sketch the graphs that represent an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (1)$$

where $A \neq 0$ or $C \neq 0$, by using dynamic geometry software. We assume without the loss generality that $A \geq 0$ because equation can be multiplied through -1 . In looking for ways to sketch the graphs with the aid of dynamic geometry software, we are forced to look for simple ways that will give us insight into the conics. The presence of the xy -term makes the construction of the graph of equation (1) difficult. One way to graph equation (1) is to treat it as a quadratic in y : $Cy^2 + (Bx + E)y + (Ax^2 + Dx + F) = 0$.

Solving for y gives us

$$y = \frac{-Bx - E \pm \sqrt{(Bx + E)^2 - 4C(Ax^2 + Dx + F)}}{2C}$$

and it can be written in the form:

$$y = \frac{-Bx - E \pm \sqrt{x^2(B^2 - 4AC) + (2BE - 4CD)x + E^2 - 4CF}}{2C} \quad (2)$$

The preparation of a table of values for x and y is a tedious task. It seems easier to use dynamic geometry software, in this case Geometer's Sketchpad, to graph equation (2), but this approach does not give us mathematical insight into the kind

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of graphs, into the role of the coefficients of equation (1), or into the rotations and translation of conics.

2. Geometrical construction with Geometer's Sketchpad

The general quadratic equation in two variables represents all the possible conics (conic sections), non-degenerate and degenerate [1]. The non-degenerate conics are called the ellipse, hyperbola and parabola. This article focuses on these non-degenerate conics; that is when $4ACF + BED - CD^2 - FB^2 - AE^2 \neq 0$. The non-degenerate conics have two axes; if these axes are identical to the coordinate axes, then the conic section is said to be in standard position. A conic section in standard position has a particularly simple equation in the case of non-degenerate conics. In these cases equation (1) can be simplified to one of the following forms by a translation and rotation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{in the case of an ellipse}) \quad (3)$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (\text{in the case of a hyperbola}) \quad (4)$$

$$y^2 = kx \quad (\text{in the case of a parabola}) \quad (5)$$

where a , b and k are constants. Although Geometer's Sketchpad makes provision for the plotting of functions, the problem is that it is not possible to translate or rotate plotted functions in this program.

A geometrical approach to the graphing of a quadratic equation is discussed in [2]. The preciseness of dynamic geometry software makes this approach even more useful. The first step in this geometrical approach is to construct a circle by using the coefficients of the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. The circle is called the *conic circle* and it goes through the points $P(A; B/2)$ and $Q(C; -B/2)$, with segment PQ as its diameter, as is illustrated in figure 1.

This can be done in Geometer's Sketchpad by plotting the points P and Q , the segment PQ , the midpoint of segment PQ , and the circle with the midpoint of segment PQ through P and Q . If the constructed circle in figure 1 does not intersect

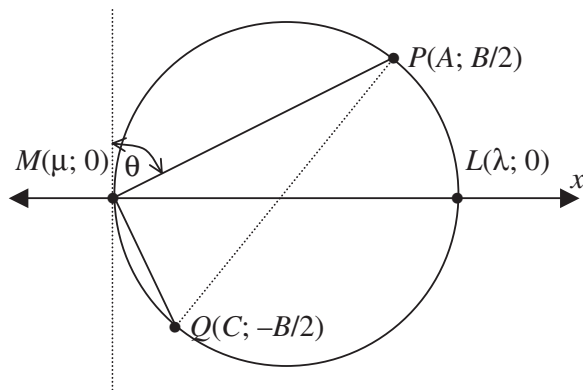


Figure 1. Conic circle.

the y -axis, the graph will be an ellipse; if it passes through the origin, it is a parabola; and if it intersects the axis in two points, it is a hyperbola. The mathematical explanation for this procedure is described by [2]. The circle intersects the axis in two points, $L(\lambda; 0)$ and $M(\mu; 0)$, where $\mu < \lambda$. These values are used to determine the equations of the translated and rotated conics. In the case of the parabola, it is

$$x^2 = \frac{-\hat{E}}{\lambda} y \tag{6}$$

and in the case of an ellipse or hyperbola, it is

$$\lambda x^2 + \mu y^2 + \tilde{F} = 0 \tag{7}$$

\hat{E} , λ and μ can be read off the graph by using Geometer's Sketchpad. Although \tilde{F} can be determined by construction, it is more efficient to calculate it numerically: $\tilde{F} = Ah^2 + Bhk + Ck^2 + Dh + Ek + F$. To determine equation (1) and the value of \tilde{F} , we need the values of \hat{E} , h and k . It is possible to graph equations (6) or (7) if these values are known, but this graph must be translated and/or rotated to represent equation (1). To determine the required rotation, we can determine the angle θ in figure 1. To find the value of \hat{E} in equation (6), construct \hat{x} -axis $\parallel MP$, \hat{y} -axis $\parallel MQ$, point $K(D/2; E/2)$ and construct the perpendicular from K on the \hat{y} -axis; this value is $\hat{E}/2$. The values of h and k are used to determine the translation and these values can be determined by finding the intersection $(h; k)$ of the lines

$$y = -\frac{2A}{B}x - \frac{D}{B} \text{ and } y = -\frac{B}{2C}x - \frac{E}{2C}$$

In summary, the following is known about the graph of equation (1): it is a conic and congruent to the graph of the simplified equations (3), (4) or (5). A translation of a distance h in the direction of the x -axis and a distance k in the direction of the y -axis is needed, as well as a rotation of θ to represent the original equation (1). To draw the graph of the original equation (1), it is necessary first to plot the graph of equation (6) or (7), depending on what kind of graph it is. Equation (6) can be written in the form

$$f(x) = \pm \sqrt{\frac{-\lambda x^2 - \tilde{F}}{\mu}}$$

and equation (7) in the form

$$g(x) = \frac{-\lambda}{\hat{E}} x^2$$

This can be done by using the plotting function in Geometer's Sketchpad. The problem, however, is that Geometer's Sketchpad does not allow us to rotate or translate plotted functions.

3. The construction of a conic

To overcome this limitation in Geometer's Sketchpad, construct five arbitrary points on the graph. The reason for the five points (no three of which are collinear) is that five points uniquely determine a conic [1]. It is possible to translate and rotate the five points using Geometer's Sketchpad. After the translation and rotation of the points,

the problem is to sketch the conic through these points. Pascal's theorem can be used to construct the conic through the five points. Pascal discovered this theorem in 1640 when he was seventeen [3]. Pascal's theorem states that if a hexagon ($ABCDEF$) is inscribed in a conic, then opposite sides intersect in three points ($AB.DE$, $BC.EF$, $CD.FA$) which are collinear (see figure 2).

Cederberg [3] and Kimberling [4] explain a way to construct additional points from five given points, using Pascal's theorem.

Let $A, B, C, D,$ and E be the five given points of a conic. The sixth point, F , can be described as the sixth point of the inscribed hexagon $ABCDEF$. The diagonal points are P (the intersection of AB and DE), Q (the intersection of BC with EF) and R (the intersection of CD with FA). We know that P, Q and R will be collinear. Draw an arbitrary line m through E and construct the points P and Q , then $R = CD \cap PQ$ and $F = RA \cap m$ (see figure 3).

To locate another point of the conic, another line must be drawn through point E . Because of the dynamic nature of Geometer's Sketchpad, this can be done by drawing a circle with E as the centre and choosing an arbitrary point V on the circle, so that m is the line EV . The animation of point V on the circle will result in the generation of the conic $F = RA \cap m$ by the tracing point F . Instead of tracing the

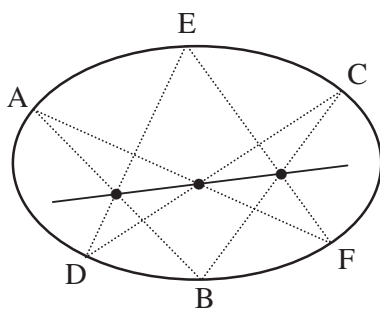


Figure 2. Pascal's theorem.

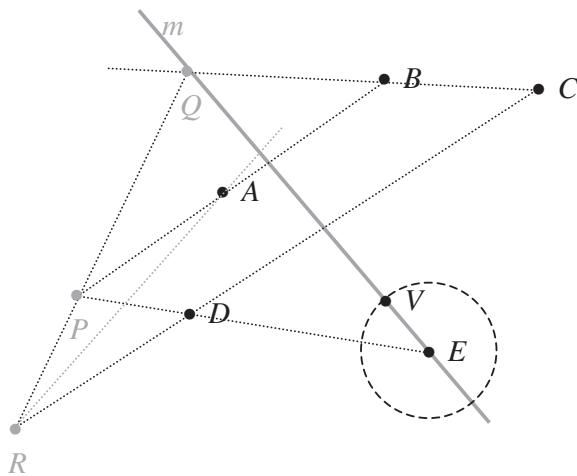


Figure 3. Construction of additional points from five given points.

point, the locus function can also be used in Geometer’s Sketchpad. At first, it seems to be a tedious process, but this construction can be recorded as a tool Geometer’s Sketchpad may provide in future. With the help of this tool the construction will be done by selecting the five points.

4. Example and summary

The whole construction process of the conics is illustrated and summarized by the following example:

Construct the graph of the equation:

$$2x^2 - 2xy + y^2 + 8x - 12y + 36 = 0$$

- Step 1: Construct the points $P(A; B/2)=(2; -1)$ and $Q(C; -B/2)=(1; 1)$ and the conic circle with PQ as diameter. M and L are the two x -intercepts, as is illustrated in figure 2. Use the measure function in Geometer’s Sketchpad to find the coordinates of M and L : $M(0.38; 0)$ and $L(2.62; 0)$. That means that $\mu = 0.38$ and $\lambda = 2.62$.
- Step 2: Draw the segments PM and PQ . The rotation angle is 31.717° and the conic section is an ellipse, because the conic circle does not intersect the y -axis.

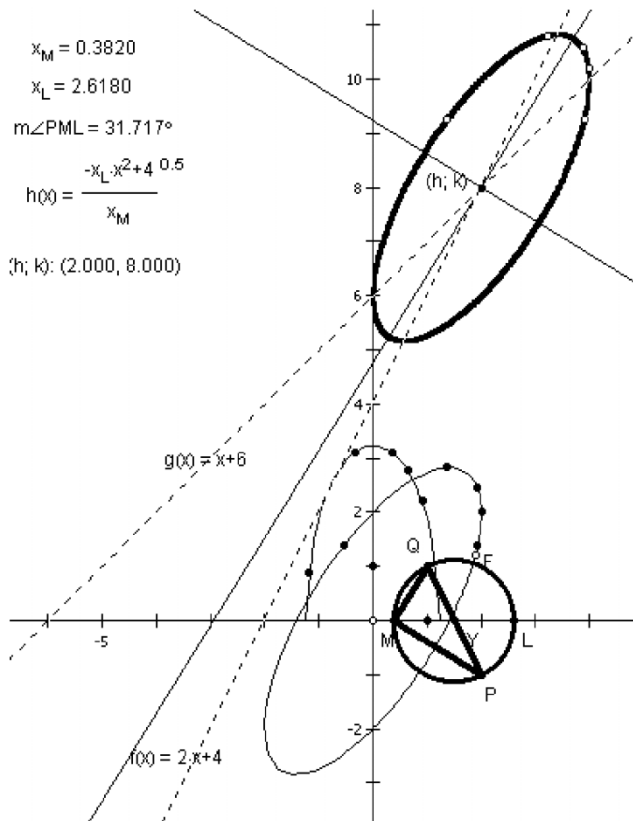


Figure 4. Geometer’s Sketchpad construction.

Step 3: Plot the lines

$$y = -\frac{2A}{B}x - \frac{D}{B} \text{ (in this case : } y = x + 6 \text{) and}$$

$$y = -\frac{B}{2C}x - \frac{E}{2C} \text{ (in this case : } y = 2x + 4 \text{).}$$

The point of intersection of the lines is $(h; k) = (2; 8)$.

Step 4: Construct \hat{x} -axis MP , \hat{y} -axis MQ through $(h; k) = (2; 8)$.

Step 5: Graph any part of the ellipse $2.62x^2 + 0.38y^2 + 4 = 0$ ($\lambda x^2 + \mu y^2 + \tilde{F} = 0$).

In figure 4 the positive part of the graph of

$$y = \pm \sqrt{\frac{-2.62x^2 + 4}{0.38}}$$

is drawn.

Step 6: Construct five points on the ellipse and rotate them, with the origin as centre, at an angle of 31.717° . Translate these points 2 units in the direction of the positive x -axis and 8 units in the direction of the positive y -axis.

Step 7: Use Pascal's theorem to construct the unique conic through these five points (see figure 4).

Whereas the appropriate rotation of conics is rather difficult to deal with computationally, it can be accomplished geometrically in a simple way by using Geometer's Sketchpad, Pascal's theorem and the graphing procedure of DeTemple [2].

Acknowledgments

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