Connecting loci with real life

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Loci is a difficult section of the syllabus, because it is taught in an algebraic way. In the algebraic approach, the main focus of teaching is to explain to the learners a way to determine the relationship between *x* and *y* in the form of an equation. Hawking (1999) believes that equations are the boring part of mathematics. He attempts to see things in terms of geometry. The aim of this paper is to approach loci geometrically by using graphs. After visualising the problem, we are going to use analytical geometry to describe the problem algebraically. By doing it this way, we can make loci more understandable, and also more enjoyable. Learners must experience the power of analytical geometry.

The teaching of loci

Locus is a Latin word, which means “place” (Guardian Education Interactive Ltd:2002). When we talk about a locus, we are talking about a path traced out by a point (or many points) as it moves according to a certain rule. An example of loci can be found in long-exposure photography, where one uses a slow shutter speed in order to blur motion purposely. This is a simple technique that can fill city streets with streaks of light or fill the night skies with skinny circles.

One way to teach loci is to do it directly, algebraically. In this approach the main objective is to determine the relationship between *x* and *y* in the form of an equation. Visualisation does not play an important role in this approach. It is not only learners that find this approach difficult. In a survey that I have done about the confidence level of teachers teaching grade 10 to 12 topics (conducted among students who enrolled for a Unisa course), one of the topics that stood out because of the low confidence level was “loci”. If so many teachers experience difficulty with it, why teach loci? The reason is that loci give learners the opportunity to describe a geometrical situation algebraically by using analytical geometry. An important role of analytical geometry, according to Moise (1990:249), is to bridge the gap between algebra and geometry: “… (the) greatest contribution to mathematics was the discovery of the coordinate systems and their application to problems of geometry. Ever since then algebra and geometry have worked together, to the advantage of both. Fishback (1969:71) also explains why it is important to move from geometry to algebra: “This introduction of coordinates achieves a fusion of geometry and algebra... In analytical geometry we are able to discover and prove geometric facts by algebraic processes, these processes frequently leading to proofs simpler than the synthetic geometric ones.” The advantage of an algebraic process is that as soon as a situation is described in algebraic terms, simple algebraic techniques can be used to solve the geometric problem (Resnikoff & Wells 1973:202). It is important for learners to experience the process of using analytical geometry to describe a geometric situation algebraically. Sibley (1998:65) defines a locus problem as “(t)he process of finding a set of points or its equation from a geometric characterization…” That means the answer to a locus problem can be either an equation or a graph (set of points) and that geometry is the origin of a locus problem. Serra (1997:122) stressed the visual aspect of loci, and explained loci as follows:

When we use visual thinking in geometry to describe points that satisfy a set of conditions or the path of something moving according to a given set of instructions, the set of points or the path is called a locus of points.

This papers explores a way of teaching loci by starting with a visual, geometric approach, because it promotes understanding. After understanding and visualising a locus, we are going to use analytical geometry to describe the situation algebraically in the form of an equation.

Exploring and drawing loci

Without knowing it, all of us have encountered the idea of a locus:

If you ask a taxi driver to take you from the conference to the beachfront and the car has an oil leakage, it is easy to visualise and describe the path of the car, looking at the oil dash.

In fact, the oil dash is a locus, because it is the path of something moving according to a given set of instructions.

We start off with a few activities.

Activity 1: *You want to train a horse. The horse is tied to a thong, which is tied to a post. What kind of graph does the galloping horse form?*

Activity 2A: *There is a lion and a river with crocodiles on a game farm (see figure 1). You have to help a friend find the safest path to walk from A to B. That is the path that is equidistant from the lion and the crocodiles in the river. What kind of graph does this path form?*

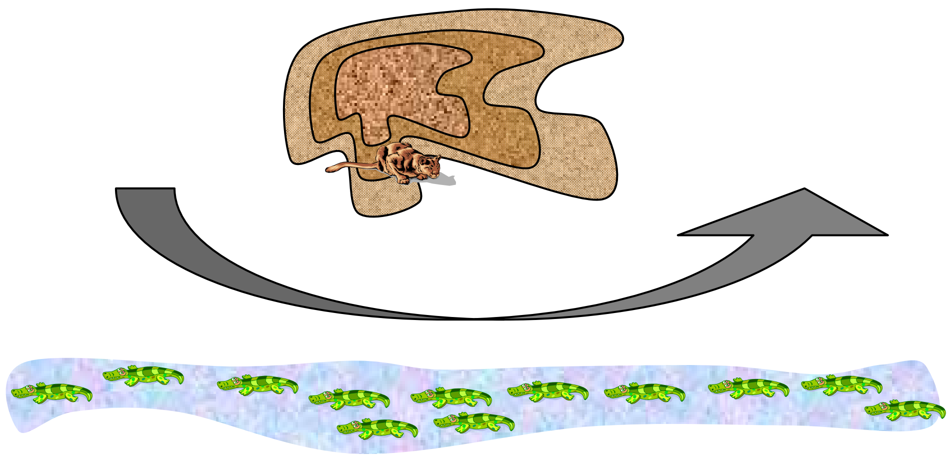


Figure 1

Activity2B: *What if the person is more afraid of the lion and wishes to walk twice as far from the lion as from the river?*

Activity2C: *What if the person is more afraid of the crocodiles and wishes to walk twice as far from the river as from the lion?*

The solutions to the above three problems are illustrated in figure 2.

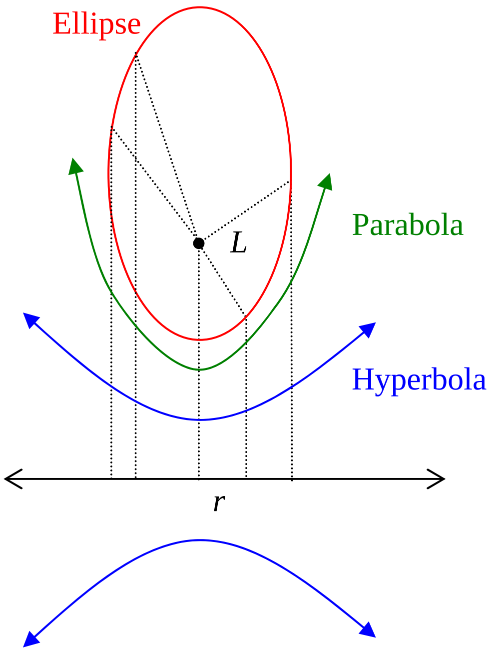


Figure 2

Activity 3: *A band of pirates left a message indicating the location of treasures they had buried (see figure 3). It read:*

Treasure A: "10 meters from the seashore and as near as possible to the light tower."
Treasure B: "As far from the light tower as from the tree AND equidistant from the tree and the seashore."

Can you find *the spots where the treasures were buried?*

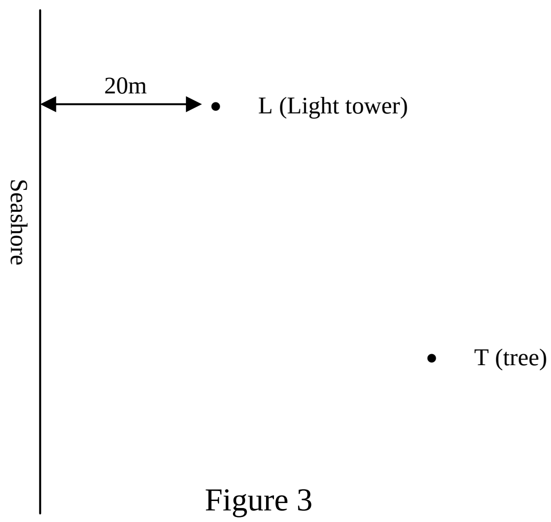


Figure 3

When working with loci questions, you need to be able to visualise the situation to find the locus. Try to picture it in your mind.

Activity 4A: *A bicycle rider fits a reflector on the spokes of the bicycle wheel, because of the danger of the traffic from the sides. Draw the motion of the reflector when the bicycle is in motion.*

Activity 4B: *Complete the locus of point P as you roll it along a surface (see figure 4).*

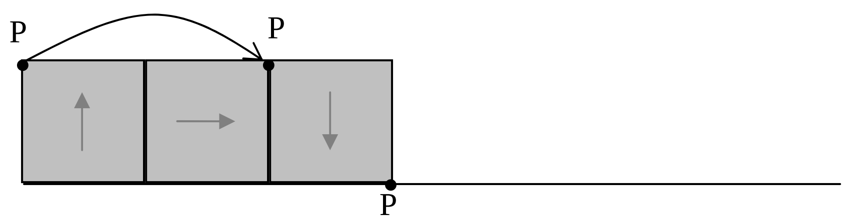


Figure 4

Finding the equation of a locus

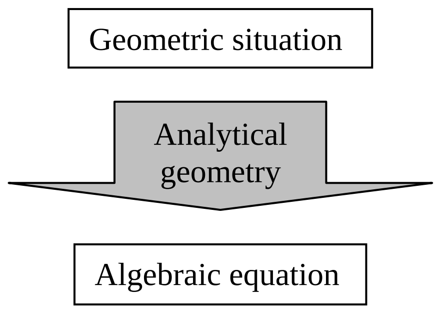
In the above activities you have done visual thinking to describe the path of something moving according to a given set of instructions, or you have described points that satisfy a set of conditions. We call that a locus of points. There are two definitions of a locus:

A locus is a set of points, which satisfy a certain condition.

or informally:

A locus is the path along which the point can move under certain conditions.

We have seen that a locus constituted of a set of points is normally a regular figure, such as a line, parabola, circle etc. But is it possible to find the equation of the locus? We know that we can use analytical geometry to describe a geometric situation algebraically. If we determine the equation, we actually find the relationship between x and y. That is the equation of a graph.



Examples of equations of loci are

- $x + y = 2$ or $2x + 3y = 6$
- $x^2 + y^2 = 16$ or $(x - 2)^2 + (y + 3)^2 = 16$
- $y = 2x^2 + 3x - 4$
- xy = 8
- $2x^2 + 3xy = y^2$

To make it easier to determine the equation of a locus, first try to draw the locus. Try to identify the kind of graph. Is it possible to determine the equation directly?

Activity 5: *Find the equation of the locus of P if P is equidistant from A(0; 3) and B(2; –1) (see figure 5).*

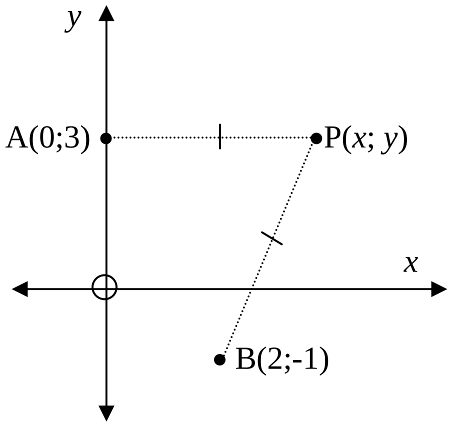


Figure 5

Try the following:

Step 1: Make a sketch (only of the situation)
Step 2: Let P(x; y) be any point on the locus.
Step 3: Write down the condition which must be satisfied, using analytical methods.
Step 4: Manipulate the equation into a useful form.

Condition: AP = PB

$$\sqrt{(x_A - x_P)^2 + (y_A - y_P)^2} = \sqrt{(x_P - x_B)^2 + (y_P - y_B)^2}$$

$$\therefore \sqrt{(0 - x)^2 + (3 - y)^2} = \sqrt{(x - 2)^2 + (y + 1)^2}$$

$$\therefore x^2 + 9 - 6y + y^2 = x^2 - 4x + 4 + y^2 + 2y + 1$$

$$\therefore 0 = 8y - 4x - 4 \quad \text{that is} \quad 0 = 2y - x - 1$$

Can you find the equation of this locus using another method?

Activity 6: *Find the equation of the locus of point P(x; y) if P is 3 units from the point A(–1; 2). Try to use two methods.*

We know is going to be a circle. The general formula for a circle is $(x - a)^2 + (y - b)^2 = r^2$ where (a; b) is the centre and r the radius of the circle. In this case the radius is 3 and the centre is the point with coordinates (–1; 2), therefore the equation of the locus is $(x + 1)^2 + (y - 2)^2 = 3^2$. Another way to do it is to describe the condition, that is AP = 3 and then apply the distance formula.

Activity 7A: *Find the equation of the locus of point P(x; y) if P is equidistant from point A(3; –4) and the straight line y = –2 (see figure 6). Find two methods!*

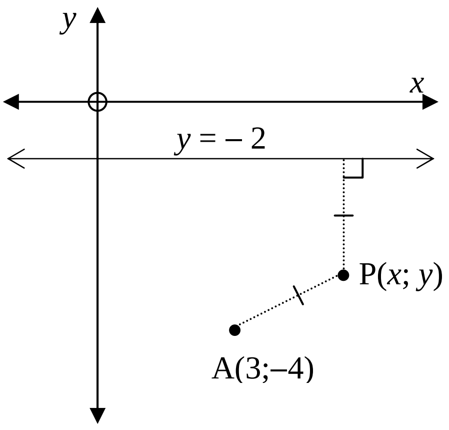


Figure 6

Activity7B: *What if P is always twice the distance from point A that it is from the line y = –2?*

Activity7C: *What if P is always twice the distance from the line y = –2 that it is from point A?*

One solution to activity 7A is:
Let Q be any point on the line $y = -2$.
Then the coordinates of Q are $(x; -2)$.
The condition:

$$PQ = AP$$
$$\therefore\ PQ^2 = AP^2$$
$$\therefore\ (y+2)^2 = (x-3)^2 + (y+4)^2$$
$$\therefore\ y^2 + 4y + 4 = x^2 - 6x + 9 + y^2 + 8y + 16$$
$$\therefore\ -4y = x^2 - 6x + 21$$
$$\therefore\ y = -\frac{1}{4}x^2 + \frac{3}{2}x - \frac{21}{4}$$

What kind of graph is represented by this equation?

Activity 8: *A fishing trailer X must pass a rocky area. There are two light towers A and B (see figure 7). The captain knows that if the angle AXB is equal to 90° they are safe. What kind of graph will the locus of X be?*
Determine the equation of the locus of X if A and B are respectively (–2; –1) and (3; –6).

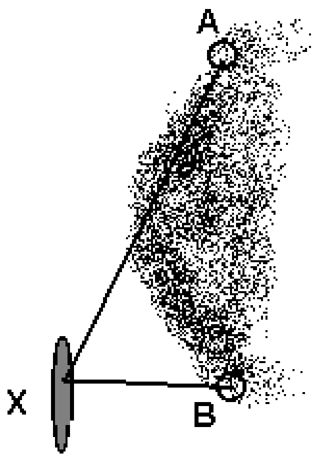


Figure 7

There are two ways to do activity 8:

Method 1

Centre of circle: $O\left(\frac{-2+3}{2};\frac{-1-6}{2}\right)$ $\therefore$ O(½; –3 ½)

Radius = ½ AB = $\frac{1}{2}\sqrt{(-2-3)^2+(-1+6)^2}$ = $\frac{1}{2}\sqrt{25+25}$ = $\frac{1}{2}\sqrt{50}$

Standard form: $(x - a)^2 + (y - b)^2 = r^2$ $\therefore$ $(x - \frac{1}{2})^2 + (y + 3\frac{1}{2})^2 = \frac{50}{4}$

Method 2 (If you did not recognise the graph)

$m_{AX}.m_{BX} = -1$

$\frac{y+1}{x+2}.\frac{y+6}{x-3} = -1$

$(y + 1)(y + 6) = -1(x + 2)(x - 3)$

$y^2 + 7y + 6 = -x^2 + x + 6$

$y^2 + 7y + x^2 - x = 0$

$y^2 + 7y + \frac{49}{4} + x^2 - x + \frac{1}{4} = \frac{49}{4} + \frac{1}{4}$

$(y + 3\frac{1}{2})^2 + (x - \frac{1}{2})^2 = \frac{50}{4}$

Summary

The direct algebraic approach that is used in our classrooms is one of the reasons learners believe that loci is difficult to understand. It is important that the learners first understand what the term locus means and then try to visualise and draw it. After drawing the picture, the learners can try to describe the graph directly in terms of *x* and *y*. By doing it this way, we make loci more understandable and more enjoyable. It is often possible to determine the equation of the locus directly. Otherwise we can use analytical geometry to describe the geometric condition algebraically in terms of *x* and *y*. This approach gives the learners a chance to experience the power of analytical geometry.

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